공학박사학위논문

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판매 촉진을 도입한 수요 불확실성 재고관리 모형

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위 원 장	박 종 헌	(인)
부위원장	문 일 경	(인)
위 원	박 건 수	(인)
위 원	이 동 호	(인)
위 원	정 병 도	(인)

Abstract

Implementation of Sales Promotions to Inventory Models with Uncertain Demand

Youngchul Shin
Department of Industrial Engineering
The Graduate School
Seoul National University

As the globalization of markets accelerates competition among companies, sales promotion, which refers to short-term incentives promoting sales of products or services, plays a prominent role. Although there are various types of sales promotions, such as price reduction, buy-x-get-y-free, and trade-in program, the common purpose is to induce the purchase of customers by offering benefits. This successful strategy has caught the attention of researchers, including operations management and supply chain management. Thus, various studies have been conducted to examine strategies for ongoing operations and to demonstrate the effects of the sales promotion, which are based on the strategic level. However, research at the tactical or operational level has been conducted insufficiently.

This dissertation examines the inventory models considering (i) markdown sale, (ii) buy one get one free (BOGO), and (iii) trade-in program. First, the newsvendor model is considered. By introducing the decision variable, which represents the start time of markdown sale, the retailer can obtain the optimal combination of the start time of a markdown sale and an order quantity. Under certain conditions in a decentralized system, however, the start time of a markdown sale where the retailer obtains the highest profit is the least profitable for the manufacturer. To avoid irrational ordering behavior by a retailer against a manufacturer, a revenue-sharing contract is proposed. Second, the mobile application, "My Own Refrigerator", is considered in the inventory model. It enables customers to store BOGO products in their virtual storage for later use. That is, customers can drop by the store to pick up the extra freebies in the future. The promotion involves a high degree of uncertainty regarding the revisiting date because customers who buy the product do not need to take both products on the day of purchase. To deal with this uncertainty, we propose a robust multiperiod inventory model by addressing the approximation of a multistage stochastic optimization model. Third, the trade-in program is considered. It is one of the sales promotions that companies collect used old-generation products from customers and provide them with new-generation products at a discount price. It also helps to acquire the additional products which are required for the refurbishment service. A multiperiod stochastic inventory model based on the closed-loop supply chain system is proposed by incorporating the trade-in program and refurbishment service simultaneously. The stochastic optimization model is approximated to the robust counterpart, which features a deterministic second-order cone program.

Keywords: Sales promotion, Newsvendor model, Inventory model, Revenue-sharing contract, Robust optimization, Distributionally robust optimization

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Chapter 1

Introduction

1.1 Sales promotion

As the globalization of markets accelerates competition among companies, *sales promotion*, which refers to short-term incentives promoting the sale of a product or service, plays a prominent role. By stimulating the purchase of a product, companies can attract new customers, hold present customers, and respond to competitors. Although there are various types of sales promotions, such as price reduction, mobile coupon, free shipping, and buy-x-get-y-free, the main purpose is to induce the purchase of customers and amplify the demand by giving benefits for customers. This successful strategy has caught the attention of researchers, including the area of management science, operations management, and supply chain management. Thus, various studies have been conducted to examine strategies for ongoing operations and to demonstrate the effects of sales promotion.

The company's decision with the introduction of the sales promotion can be divided into three-level, which are strategic, tactical, and operational levels. Strategic-level decisions are made for long-term planning purposes, such as with network-design or contracts of the supply chain. Tactical-level decisions are made for mid-term planning purposes, such as with inventory policies. Operational-level decisions are made for short-term planning purposes, such as focusing on scheduling, lot-sizing, or routing in the manufacturing or remanufacturing plant.

Extant literature on the sales promotion to the inventory problem mainly focused on investigating the effects of the sales promotion ([81, 131]) or finding the optimal level of benefit or decision, which is based on the strategic-level ([29, 50, 94]). The decision on the tactical-level or operational level was not extensively studied. For retailers at the tactical or operational level, types of decisions, uncertainty, and complexity of the model could be increased. That is, a new method for inventory control is required. This dissertation focuses on the inventory management problem when the sales promotion is introduced. In detail, this dissertation examines the three types of inventory models considering (i) the markdown sale, (ii) buy one get one free (BOGO), and (iii) trade-in program. Although three different sales promotions are considered, these promotions have in common that they induce purchases of customers by providing benefits. Also, uncertain demands are considered in the three types of inventory models. In other words, three different inventory models are studied independently, but they are closely related in terms of the implementation of sales promotion and uncertain demand from the inventory management perspective.

1.2 Inventory management

The inventory management is becoming increasingly critical to the retailer. In particular, there has been a growing interest in inventory policies that respond to demand uncertainty. The method of inventory control to deal with the uncertain demand is mainly divided into the *stochastic optimization model*, *robust optimization model*, and *distributionally robut optimization model*. In the case of the stochastic optimization model, the probability distribution of uncertain data is assumed to be known or can be estimated ([93]). The decision maker seeks the optimal solution that minimizes or maximizes the expectation of the objective function under a given or estimated probability distribution.

On the other hand, the robust optimization model assumes that uncertain data belongs to a specific uncertain set instead of estimating the probability distribution. Unlike the stochastic optimization model that makes decisions over the distribution, it finds the optimal solution of the worst-case scenario, which could be feasible for all possible scenarios. The robust counterpart of the optimization model can maintain the tractability from the primal deterministic optimization model and has the advantage of being solvable without estimation of the full information about distribution. In other words, the model requires partial information of the uncertain factors or random variables. However, because of the conservative solution, it provides a worse solution than the stochastic optimization model. In contrast to the robust optimization model, the tractability of the stochastic optimization model is restricted except for some situations that satisfy certain conditions. In general, tractability is not guaranteed in a multistage decision process.

Meanwhile, the distributionally robust optimization model generalizes the stochastic optimization model and robust optimization model. By considering the ambiguity set, which contains the true distribution, but the distribution is not known, the decision maker seeks the optimal solution under the worst-case distribution or worst-case expectation. If the candidate distributions in the ambiguity set contain only the true distribution, it becomes the stochastic optimization model. On the contrary, if the candidate distributions consider all distributions under the given support set, it becomes the robust optimization model ([80]). In other words, the distributionally robust optimization is a general form of the stochastic optimization and robust optimization approaches. Accordingly, this approach provides less conservative solution than the robust optimization, while retaining the tractability from the primal deterministic optimization model. In Chapter 2, the newsvendor model based on the stochastic optimization model, is considered. In the cases of Chapters 3 and 4, robust optimization and distributionally robust optimization approaches are utilized to the multiperiod inventory models.

Research on an inventory problem that determines the order quantity and price by developing the demand function in a price-dependent form has been actively conducted. Beginning with Whitin [112], Zabel [125] and Young [124] developed price-dependent demand functions in inventory problems. After that, Lau and Lau [59] introduced the concept of pricing in the newsvendor model. By combining the pricing in the newsvendor model, the *price-setting newsvendor model* was defined and has been studied actively ([17, 40, 52, 58, 76, 78, 83, 87, 113, 114, 118, 120]). By utilizing the price-setting newsvendor model, various research considering the markdown has also been studied ([67, 74, 96]). However, another important issue has been overlooked: the timing of a markdown sale. In contrast with many studies concentrated on the pricing and order quantity in the inventory model, this dissertation focuses on an optimal combination of the start time of the sale and the order quantity under the predetermined discount rate.

There exist several previous studies which are relevant to the multiperiod inventory model based on the robust optimization. Bertsimas and Thiele [15] applied a robust optimization based on a polyhedral uncertainty set to the multiperiod inventory model. Ben-Tal et al. [8] adapted the adjustable robust optimization approach to the retailer-supplier flexible commitment contract, which is the expanded form of the multiperiod inventory model. See and Sim [88] considered the multiperiod inventory problem whose objective function is presented as the expectation under stochastic demand. They considered stochastic demand as a factor-based demand model that is affinely dependent on the primitive uncertainty factors. By adopting a linear decision rule and utilizing the distributionally robust bound, which is presented by Chen and Sim [26], the multistage stochastic inventory model was derived to a second-order cone optimization model. Meanwhile, Ang et al. [3] studied a storage assignment problem. They also considered a stochastic demand as a factor-based demand model and derived the tractable formulation by utilizing the linear decision rule.

1.3 Research motivations

The implementation of sales promotion increases the uncertainty and complexity of the decision to the inventory manager of the retailer. It addresses that a new method of inventory control is required. This dissertation aims to provide the proper inventory model and derive the managerial insights that could be beneficial to the inventory manager.

In the case of the markdown sale, the determination of an appropriate start time of price reduction could remove the unnecessary inventory while maximizing the revenue. The importance of inventory management for perishable items has been steadily attracting attention. Because of the characteristics of items whose values drop precipitously or cannot be sold after a particular time, items should be disposed of by a markdown sale. Extant literature on the inventory problem mainly focused on investigating decisions on selecting products for a discount or the amount of the discount. That is, the decision on the start time of the markdown sale was not extensively studied. This dissertation focuses on the optimal combination of a start time of the markdown sale and an order quantity based on a newsvendor model. Under certain conditions in a decentralized system, the start time of a markdown sale where the retailer obtains the highest profit is the least profitable for the manufacturer. Therefore, we propose a revenue-sharing contract to avoid irrational ordering behavior by a retailer against a manufacturer.

In the case of the research considering the BOGO promotion, it was motivated by a mobile application "My Own Refrigerator". This mobile application enables customers to store BOGO products in their virtual storage for later use. That is, customers who store the extra freebies in their virtual storage can drop by the store to pick them up in the future. Consequently, the application was successful in attracting customers. However, this type of promotion has significant implications for inventory levels. Since customers who buy the product do not need to take both products on the day of purchase, the promotion involves a high degree of uncertainty regarding the revisiting date. To deal with this uncertainty, we propose a distributionally robust multiperiod inventory model by addressing the approximation of a multistage stochastic optimization model.

For the research considering the trade-in program, it was motivated by the companies selling high-tech products, such as *Apple* or *Samsung*. The trade-in program is one of the sales promotions that companies collect used old-generation products from customers and provide them with new-generation products at a discount price. It has led customers to pursue repeating purchases of new-generation products and prevented customers from dropping out to competitors. It also helps to acquire the additional products required for the refurbishment service, which is a warranty service that provides the like-new condition of old-generation products through the refurbishing process. In this closed-loop supply chain system, the retailer should consider that customers using the trade-in program be provided with the new-generation products. Furthermore, two additional decisions related to remanufacturing and refurbishing are required for returned products through the refurbishment service and trade-in program, respectively. Accordingly, we propose a multiperiod inventory model based on the closed-loop supply chain system that incorporates the refurbishment service and trade-in program simultaneously.

1.4 Research contents and contributions

This dissertation aims to study the three types of inventory problems with sales promotions. First, the optimal combination of a start time of the markdown sale and an order quantity based on a newsvendor model is studied. Under certain conditions in a decentralized system, the start time of a markdown sale where the retailer obtains the highest profit is the least profitable for the manufacturer. Therefore, a revenue-sharing contract is proposed to avoid irrational ordering behavior by a retailer against a manufacturer. Centralization through the revenue-sharing contract improves the profits of the retailer and manufacturer compared to those earned in the decentralized system.

Second, the multiperiod inventory model considering the BOGO promotion is examined. To handle the uncertain revisiting rate of customers, a linear decision rule and distributionally robust bound are utilized. By approximating a multistage stochastic optimization model with the linear decision rule, the distributionally robust inventory policy can be derived without full information on the distribution, only requiring the support and the first and second moments of uncertainty factors. The presented model is different from previous studies in that the sum of the uncertainty factors in a particular interval is constrained to less than or equal to 1. This part is reformulated as a robust counterpart that retains tractability under modest data sizes. The results of the comparative simulation experiments show that the presented model provides a robust solution against the worst-case distribution. We also obtain managerial insights from the experiments by varying the expiry date according to three types of customers' revisiting tendencies.

Third, the multiperiod closed-loop supply chain system considering the refurbishment service and trade-in program is proposed. Three types of correlated uncertain demands, which are for the new-generation product, refurbishment service, and trade-in program, are considered. By adopting the factor-based demand model, which is affinely dependent on the predefined uncertain factors, correlations among these uncertain demands are characterized. To retain the tractability of the model, we approximate the multistage stochastic optimization model to the affinely adjustable robust counterpart, which features a second-order cone program. Computational results provide managerial insights that could be beneficial to the inventory manager.

1.5 Outline of the dissertation

In this dissertation, three types of inventory problems with sales promotions, including the markdown sale, BOGO, and trade-in program, respectively, are studied. In Chapter 2, we consider the optimal start time of the markdown sale in the newsvendor model. To prevent the irrational order from the retailer, a revenue-sharing contract is proposed. In Chapter 3, we consider the multiperiod inventory model with the mobile application "My Own Refrigerator." To deal with the uncertain revisiting rate of the customers, we adopted the distributionally robust optimization approach. In Chapter 4, we consider the trade-in program in the closed-loop supply chain. By incorporating the correlated uncertain demands in the multiperiod inventory model, we adopt the factor-based demand model. To derive the tractable formulation, we utilized the linear decision rule and distributionally robust bound. In Chapter 5, we summarize the findings of this research and suggest future research.

Throughout the dissertation, we will use the bold characters to denote vectors, such as $\mathbf{x}$. The operators $(\cdot)^+$ and $(\cdot)^-$ mean $\max(\cdot, 0)$ and $-\min(\cdot, 0)$, respectively. Meanwhile, the tilde, such as $\tilde{x}$, represents the uncertain values.

Chapter 2

Optimal Start Time of a Markdown Sale Under a Two-Echelon Inventory System

2.1 Introduction and literature review

Inventory management on *perishable items* has been steadily attracting attention from researchers in various academic fields, including operation management, marketing, and business administration. In general, perishable items refer to products that see a precipitous drop in value or that cannot be sold after a certain time because of their finite or limited shelf life. In the past, the term was used to describe products, especially food, that decay quickly. In recent years, however, as product development life cycles have shortened and global competition has intensified, more types of products have come to be regarded as perishable items. For example, high-tech devices, such as mobile phones, are launched more often than ever before, and fast fashion goods that were used to be introduced quarterly are now released monthly or weekly. The lifespan of food in a supermarket also decreases because of an increase in customer demand for freshness ([71]). Accordingly, the traditional method running the inventory by maintaining stocks for a long period no longer confers a competitive advantage. Customers regard the products already stored in inventory for a long time as technically cluttered or stale products ([5]).

Consequently, keeping goods in stock over a long time eventually causes loss of profitability ([4]).

Various studies have been widely conducted to deal with perishable items. A newsvendor model is one of the conventional approaches used to cope with perishable items in inventory management. The model provides an optimal order quantity by considering the trade-off between overestimating and underestimating customer demand. A general assumption in the basic newsvendor model is that a retailer orders a single item from a manufacturer (supplier) by determining the optimal order quantity to meet the uncertain demand within a single-period. This classical model has been extended to various ways ([53, 79]). Although various extensions were developed, a fluctuation of the price within a single-period was not considered. Even though the newsvendor model was extended to the multi-period model, it recursively solved the problem based on the single-period model. In the case of the perishable item, it would be worth noting by expressing the price fluctuation during a single-period to illustrate the last order situation at the end of the selling season.

For perishable items near the end of the selling season, the company might earn more profit by selling all of the remaining stocks with the lower price rather than disposing of the entire leftover stocks. Outdated stocks not only hinder the flow of capital but also occupy the space used for a new product. In addition, relatively old products lose competitiveness because of the new entry of competitive products into the market. Furthermore, a company selling an out-of-style item at a low price can degrade the brand image. According to *The Times* magazine, *Burberry*, which is the luxury brand, incinerated up to as much of £90m worth of stocks in July 2018.

To deal with these issues, many companies have introduced *pricing strategy* to reduce the loss incurred by perishable items. It refers not to passive acceptance of existing customer demand but the proactive response for amplification of demand. By reducing the price for the same product over time, more demands can be generated by attracting interest from customers who want to purchase the product at a sale price. A markdown sale is a representative example of a pricing strategy. In the case of a promotion, the price is not permanently being reduced, and it can change over time. It is related to the studies considering dynamic pricing or multiple price markdowns ([28, 42, 70, 115]). In the literature on economics, it is widely known that the price and demand are in inverse proportion to each other ([56, 116, 123]). Research on forward-looking customers, who are willing to wait for a price reduction and purchase when the price is discounted, also explains the inverse relationship between price and demand. Pesendorfer [75] claimed that customers who put a low valuation on a product expect the product to be sold at a lower price in a markdown sale. By accommodating these customers' expectations, especially in the apparel industry, the company promotes a markdown sale for over-stock items at the end of the selling season.

The area of inventory management also has shown an interest in pricing ([32]). In addition to research areas such as demand forecasting, optimal order quantity, or pricing, researchers in the inventory management have focused on determining the price and order quantity simultaneously ([67, 74, 76, 96]). By extending the newsvendor model, including the determination of the price of the product, the model provides the optimal price and order quantity at a time ([27, 34, 47]). However, another important issue has been overlooked: *timing* of a markdown sale. It may bring about the following question: *what is the optimal time to reduce the price*?

The determination of an appropriate start time of price reduction could remove the unnecessary inventory while maximizing the revenue. This information could also have a significant impact on the retailer's last order quantity, which consequently affects the profit of the manufacturer and supply chain system. Depending on the situation, a retailer might earn the maximum profit when a markdown sale starts as early as possible. In contrast, a late markdown sale generates maximum profit. In a particular case, starting a sale in the middle of the selling period leads to the maximum profit. Otherwise, the retailer is indifferent to the start time of the markdown sale. Depending on the start time of the sale, customer demand and the order quantity of the retailer from the manufacturer could vary. In other words, the profits of the retailer and manufacturer vary based on the start time of the markdown sale. Despite the importance of the decision of the start of the markdown sale, relevant research has not been conducted sufficiently. Instead, most previous studies focused on determining the price and order quantity. To emphasize the distinctive feature of this study, the relevant studies are summarized in Table 2.1.

In this chapter, we analyzed the optimal combination of the start time of a markdown sale and an order quantity to generate the maximum profit at the end of the selling season. We extended the newsvendor model to consider the start time of a markdown by dividing the single-period into two parts with (i) a regular price and (ii) a sale price. In practice, the discount rate of a markdown sale, such as 30%, 40%, or 50%, is often predetermined. In contrast with many studies concentrated on the pricing and order quantity in the inventory model, we focus on an optimal combination of the start time of the sale and the order quantity under the predetermined discount rate.

Table 2.1: Summary of the previous research relevant to the price-setting newsvendor model

Authors (year)	**Model**	**Demand function**	**Distinctive feature**
Whitin (1955) [112]	EOQ[1]	Linear form	First proposed the price decision in the inventory problem
Young (1978) [124]	Monopoly	Additive & multiplicative	Generalized the results of previous relevant studies
Lau and Lau (1988) [59]	NM[2]	Additive	First proposed the price decision in the classic newsvendor model
Polatoglu (1991) [78]	NM	Additive & multiplicative	Examined the unimodality of the expected revenue
Yao et al. (2006) [118]	NM	Price elasticity mean & increasing failure rate on random demand	Identified the unimodality or quasi-concavity of the model
Granot and Yin (2008) [40]	NM	Multiplicative	Considered the postponement strategy and contract
Xu et al. (2010) [113]	NM	Additive & multiplicative	Investigated the effects of demand uncertainty on the decision
Xu et al. (2013) [114]	NM	Multiplicative demand model and random yield rate	Considered in-house production and procurement cases
Kazaz and Webster (2015) [52]	NM	General function dependent on the price and supply	Analyzed the effect of risk aversion and the uncertainty of demand and supply on the tractability and optimal decision
Ye and Sun (2016) [120]	NM	Additive & multiplicative	Compared myopic and strategic consumers in the model
Kyparisis and Koulamas (2018) [58]	NM	Nonnegative linear additive	Identified the distribution that makes the expected profit as quasi-concave in the retailer price
Rubio and Baykal (2018) [83]	NM	Additive	Evaluated the unimodality of the risk-sensitive case by analyzing a mean–variance framework
Biswas and Avittathur (2018) [17]	NM	Additive	Considered the salvage value as a variable which is dependent on the leftover inventory
Schulte and Sachs (2020) [87]	NM	Poisson demand	Proposed the simple decision rule for the inventory policy
This research	NM	Additive	Considered the start time of the markdown sale as a decision variable

EOQ[1] and NM[2] indicate the economic order quantity and newsvendor model, respectively.

When the retailer determines the optimal start time of the markdown sale and the order quantity from an individual perspective, it may lead to a local optimum. In other words, a decision made under a decentralized system cannot achieve maximum profit from the perspective of the overall supply chain system. In this system, the optimal quantity ordered by the retailer is different from the optimal quantity that the manufacturer would like to sell. Researchers have studied supply chain contracts in an effort to determine how to prevent such a local optimum. If the contract is appropriately designed, the supply chain is coordinated to ensure that the optimal order quantities for both the retailer and manufacturer coincide. Naturally, total supply chain profit, which includes the profits of the retailer and manufacturer, can thus be maximized. In a similar manner, the start time of the markdown sale should also be considered from the perspective of supply chain coordination. The retailer and manufacturer may prefer different start times for the markdown sale under a decentralized system. Thus, the contract mechanism must be properly designed to achieve supply chain coordination in terms of the start time of the markdown sale. Therefore, we examine supply chain coordination after analyzing the decentralized system.

The remainder of this chapter is organized as follows: Section 2.2 describes the demand modeling used for this research. Specifications of the profit functions and decisions of the retailer and manufacturer under a decentralized system are addressed in Section 2.3. Section 2.4 presents the profit functions and decisions under a centralized system through a revenue-sharing contract. The findings of this research are summarized in Section 2.5.

2.2 Problem description

In this chapter, we assume that a single retailer (newsvendor) places an order to a single manufacturer (supplier) at the end of the selling season. After observing the wholesale price and other relevant costs, the retailer determines the start time of the markdown sale and the order quantity. Let $t \in [0, T]$ denote the planning horizon, where the markdown sale starts at $t = t_m$. The selling period ends at $t = T$, which is the expiration date for remaining items. The period is divided into two parts as $[0, t_m]$ and $[t_m, T]$. Until t_m, items are sold at a selling (regular) price, which is subsequently decreased with a discount rate $\alpha \in (0, 1)$ after t_m. Both the price and discount rate were exogenously determined. The order is placed at $t = 0$ and covered until T. After T, no additional profit can be earned. Table 2.2 presents a summary of the random variables representing the uncertain demands considered in this study.

The terms $y(p)$ and $y(p, \alpha)$ represent the general $price - dependent$ functions with the deterministic demand where the discount rate α serves to lower the price and increase the demand. These functions represent the expected demands in the planning horizon. We adopted the additive demand function, where $y(p) = a - bp$ and $y(p, \alpha) = a - (1 - \alpha)bp$. The notation ϵ incorporates a $price - independent$ random variable, which denotes the demand uncertainty. We assume that both random terms, indicated by ϵ in the ξ and ξ', are independent and identically distributed (IID). It should be noted that the random variables representing the demands are D and D', instead of using ξ and ξ'. D and D' represent the random variables following the uncertain demand in $[0, t_m]$ and $[t_m, T]$, respectively. The demands D and D' are expressed as a linear combination of ξ and ξ' with the ratio for each sale period in the planning horizon. Figure 2.1 illustrates the random variables representing the

Table 2.2: Random variables representing uncertain demands considered in this study

$\xi = y(p) + \epsilon$	Demand in $[0, T]$ when the item is sold at the regular price
$\xi' = y(p, \alpha) + \epsilon$	Demand in $[0, T]$ when the item is sold at the sale price
$D = \frac{t_m}{T}\xi = \frac{t_m}{T}y(p) + \frac{t_m}{T}\epsilon$	Demand in $[0, t_m]$ when the item is sold at the regular price
$D' = \left(\frac{T-t_m}{T}\right)\xi' = \frac{T-t_m}{T}y(p, \alpha) + \frac{T-t_m}{T}\epsilon$	Demand in $[t_m, T]$ when the item is sold at the sale price
$\epsilon \sim N(0, \sigma^2)$	Random variable following a normal distribution f as probability density function and F as a cumulative distribution function

uncertain demands and the three possible situations for the inventory level in the planning horizon $[0, T]$.

We assume that the total demand in the planning horizon is controllable by changing the start time of the markdown sale, t_m. If the selling period with the regular price becomes longer, t_m becomes larger in D and $\left(\frac{T-t_m}{T}\right)$ becomes smaller in D'. That is, demand more increases if the period of price reduction is extended longer. Conversely, demand more decreases if the period of price reduction is shortened. To support that there is no major contradiction in the assumption, limits for t_m as to zero and T are described in Equations (2.1) and (2.2), respectively.

$$\lim_{t_m \to T-} D = \lim_{t_m \to T-} \frac{t_m}{T}\xi = \lim_{t_m \to T-} \left(\frac{t_m}{T}y(p) + \frac{t_m}{T}\epsilon\right) = y(p) + \epsilon = \xi \tag{2.1}$$

$$\lim_{t_m \to 0+} D' = \lim_{t_m \to 0+} \left(\frac{T - t_m}{T}\right)\xi' = \lim_{t_m \to 0+} \left(\frac{T - t_m}{T}y(p, \alpha) + \frac{T - t_m}{T}\epsilon\right) = y(p, \alpha) + \epsilon = \xi' \tag{2.2}$$

As shown in Equation (2.1), when t_m reaches T, which means the product is sold at the selling price p for the entire period $[0, T]$, the demand is equivalent to ξ. Similarly, by Equation (2.2), when t_m approaches 0, which means the product is sold at the sale price $(1 - \alpha)\,p$ for the entire period, the demand function becomes equal to ξ'. It is reasonable that the mean of each demand is proportional to the remaining duration

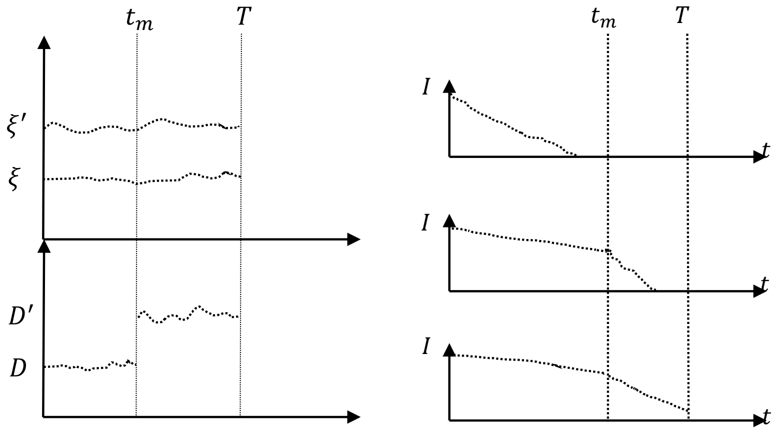


Figure 2.1: Uncertain demands considered in this study and three types of inventory levels

for each sale in D and D', respectively. As shown in Table 2.2, the variances are also proportional to the duration of the sales in D and D'. Conceptually, ϵ expresses the uncertainty of the demand, indicating that demand cannot be accurately predicted due to external or internal factors. In other words, the variance of the demand caused by uncertainty may increase when the remaining selling period is extended. Accordingly, the variance of demand is expressed as a product of the variance ϵ and the remaining selling period. For a general expression, ϵ can be used as a different random variable instead of an IID, but two reasons support the argument for setting it as an IID. First, by setting ϵ as an IID, the difference of variance can be affected solely by the remaining period rather than other factors. The main objective of our study is to analyze how the retailer's order quantity varies depending on the length of the remaining selling period. Therefore, we made the variance dependent only on the remaining period. Second, for ease of analysis, the two random variables are set to IID. Setting the random variables in different manners makes it challenging to deal with the expected profit function. Consequently, the analysis becomes difficult and the interpretation may not be intuitive. Therefore, we assume variances in demand functions as IID.

2.3 Analysis of the decentralized system

We analyzed a decentralized system in which a retailer and manufacturer consider the profit maximization from their respective positions. The retailer determines the start time of the markdown sale and the order quantity for the profit maximization within a given parameter. Meanwhile, the profit of the manufacturer depends on the order quantity determined by the retailer. Denote by q^* as the optimal order quantity when the start time of the markdown sale t_m is given. The term t_m^* indicates the optimal start time of the markdown sale when the order quantity q is given. The optimal combination for maximizing the expected profit function of the retailer is defined as (t_m^{**}, q^{**}). A newsvendor model is introduced to incorporate the expected profit function of the retailer.

2.3.1 Newsvendor model for a retailer

An objective function of a retailer is to maximize the total expected profit at the end of the selling season. Let $R_1(q, t_m)$, $R_2(q, t_m)$, and C defined as

$$
\begin{aligned}
&R_1(q, t_m) = p \cdot \mathbb{E}\left[\min(q, D)\right] \\
&R_2(q, t_m) = (1 - \alpha) \cdot p \cdot \mathbb{E}\left[\min\left((q - D)^+, D'\right)\right] \\
&C = c_r q + wq
\end{aligned}
$$

where $\mathbb{E}$ denotes expectation. The expected profit function of the retailer Π_r in the planning horizon $t \in [0, T]$ can be expressed as follows:

$$\begin{aligned}\Pi_r(q, t_m) &= R_1(q, t_m) + R_2(q, t_m) - C \\ &= p \cdot \mathbb{E}\left[\min(q, D)\right] + (1-\alpha) \cdot p \cdot \mathbb{E}\left[\min\left((q-D)^+, D'\right)\right] - c_r q - wq\end{aligned}$$

The expected profit on the planning horizon is the difference between the sum of the two types of revenues in $[0, t_m]$ and $[t_m, T]$, and the total ordering cost. Decision variables q and t_m are defined as the order quantity and the start time of the markdown sale, respectively, which are non-negative real variables . Without loss of generality, the lead time is not taken into account, which means the order quantity q is held in stock at time $t = 0$. The revenue R_1 in $[0, t_m]$ is described by $p \cdot \mathbb{E}[\min a(q, D)]$ for the product of the selling price p with the smaller value between the uncertain demand in $[0, t_m]$ and order quantity q. R_2 is the revenue in $[t_m, T]$ expressed as $(1-\alpha) \cdot p \cdot \mathbb{E}[\min((q-D)^+, D')]$ for the product of the sale price $(1-\alpha) \cdot p$ with the smaller value between the remaining inventories at t_m and the uncertain demand in $[t_m, T]$. The total ordering cost is expressed in $c_r q + wq$ where c_r is the retailer's per-unit cost and w is the wholesale price for a transfer payment. To avoid triviality, $c_r \in [0, p]$ was assumed.

In the existing literature on inventory management, although the salvage value was imposed on the leftover stock in common, it is not considered in this study. Although it is included in the profit function, it does not have a significant effect on the analysis. Therefore, the salvage value is not considered in this study.

Proposition 1. *The expected profit function of the retailer Π_r is strictly concave* *respect to q where $q \geq 0$ and given $t_m \in [0, T]$. Therefore, there exists a unique q^* maximizing the expected profit function Π_r when t_m is given.*

Proof 1. *See Appendix A.1.*

Proposition 2. *When an optimal order quantity $q^*(t_m)$ is given, critical ratio (fractile) $\frac{p-(c_r+w)}{p}$ can be expressed as a convex combination of $F(\frac{T}{t_m}q^* - (a-bp))$ and $F(q^* - (a-bp) - \frac{T-t_m}{T}\alpha bp)$.*

$$\alpha F\left(\frac{T}{t_m}q^* - (a-bp)\right) + (1-\alpha)F\left(q^* - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) = \frac{p-(c_r+w)}{p}$$

Proof 2. *See Appendix A.2.*

Corollary 1. *An optimal order quantity of the retailer has the lower and upper bounds shown in the following inequality.*

$$\frac{t_m}{T}F^{-1}\left(\frac{p-(c_r+w)}{p}\right) + \frac{t_m}{T}(a-bp) \leq q^* \leq F^{-1}\left(\frac{p-(c_r+w)}{p}\right) + a - bp + \frac{T-t_m}{T}\alpha bp \quad (2.3)$$

Proof. By Proposition 1, the following inequality holds true.

$$F\left(q^* - a + bp - \frac{T-t_m}{T}\alpha bp\right) \leq \frac{p-(c_r+w)}{p} \leq F\left(\frac{T}{t_m}q^* - (a-bp)\right) \quad (2.4)$$

If Inequality (2.4) is rearranged based on q^*, it is equal to Inequality (2.3). □

Also, Inequality (2.4) can be expressed with the expectations of demands D and D' as shown in Inequality (2.5). Recall that $\mathbb{E}[D] = \frac{t_m}{T}(a-bp)$ and $\mathbb{E}[D+D'] = a - bp + \frac{T-t_m}{T}\alpha bp$.

$$\frac{t_m}{T}F^{-1}\left(\frac{p-(c_r+w)}{p}\right) + \mathbb{E}[D] \leq q^* \leq F^{-1}\left(\frac{p-(c_r+w)}{p}\right) + \mathbb{E}[D+D] \quad (2.5)$$

Proof 3. *$\mathbb{E}[D+D']$ is larger than or equal to $\mathbb{E}[D]$. Also $F^{-1}\left(\frac{p-(c_r+w)}{p}\right)$ is always larger than $\frac{t_m}{T}F^{-1}\left(\frac{p-(c_r+w)}{p}\right)$ because $t_m \leq T$ and $F(\cdot)$ is the non-decreasing function.*

Although a closed-form is not proposed for q^*, lower and upper bounds of q^* are suggested. These bounds can be utilized to obtain q^* efficiently through the bi-section method. Details are given in the next subsection. We will analyze how q^* varies with changes of t_m. Depending on t_m, the optimal order quantity q^* and the expected profit vary. Accordingly, we need to analyze the profit function with respect to t_m.

Proposition 3. *The expected profit function of the retailer* Π_r *is strictly concave with respect to* t_m *where* $t_m \in [0, T]$ *and given* $q \geq 0$*. Thus, there exists a unique* t_m^* *maximizing the expected profit function* Π_r *when* q *is given.*

Proof 4. *See Appendix A.3.*

Propositions 1 and 3 show that Π_r is strictly concave with respect to q and t_m. We now show that Π_r is jointly concave with q and t_m.

Proposition 4. *The expected profit function of the retailer* Π_r *is strictly concave with respect to* q *and* t_m *where* $t_m \in [0, T]$ *and* $q \geq 0$*. There exists a unique combination* (t_m^{**}, q^{**}) *maximizing the expected profit function of the retailer* Π_r*.*

Proof 5. *See Appendix A.4.*

According to Proposition 4, the expected profit function is maximized through the optimal combination of the start time of the markdown sale and the order quantity. The solution procedure for obtaining the optimal combination is described in the next subsection.

2.3.2 Solution procedure for an optimal combination of the start time of the markdown sale and the order quantity

We applied a bi-section method to obtain the optimal combination of the start time of the markdown sale and the order quantity. The bi-section method is one of the root-finding methods. It searches a solution by repeating a procedure based on dividing an initially given interval until the value of the function is less than tolerance (TOL). The criterion for dividing the interval is whether the value of the function obtained by the middle point of the interval is positive or negative. The procedure is repeated by dividing the given interval in half and defining each half as a new interval. When the value of the function of the middle point is smaller than the tolerance, the procedure is terminated. According to Bolzano's intermediate value theorem, the bi-section method is guaranteed to converge if $h(a)$ and $h(b)$ have opposite signs where $h(\cdot)$ is a continuous function in the interval $[a, b]$ ([84]). That is, q^* can be obtained by setting the value of the first derivative of the expected profit function to zero. The initial interval was set as the lower and upper bounds of the q^* in Corollary 1. Likewise, the optimal start time of the markdown sale t_m^* can be obtained by using the initial interval $[0, T]$. In this manner, the optimal combination (t_m^{**}, q^{**}) can be obtained by iterating the procedure recursively. The pseudocode of the bi-section method algorithm is described in Algorithm 1.

2.3.3 Profit function of a manufacturer

In this study, we assume that the capacity of the manufacturer is infinite. Under the assumption, the profit of the manufacturer is proportional to q determined by the

Algorithm 1 Bi-section method algorithm

Initialization:
TOL=sufficiently small value
$LB = \frac{t_m}{T} F^{-1}\left(\frac{p-(c_r+w)}{p}\right) + \frac{t_m}{T}(a-bp)$
$UB = F^{-1}\left(\frac{p-(c_r+w)}{p}\right) + (a-bp) + \frac{T-t_m}{T}\alpha bp$
$LT = TOL$
$UT = T$
$u(g,t_m) = p - \alpha p F\left(\frac{T}{t_m}q - (a-bp)\right) - (1-\alpha)pF\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right) - (c_r+w)$
$v(g,t_m) = (a-bp)F\left(\frac{T}{t_m}q-(a-bp)\right) - (1-\alpha)bpF\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right) + \int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx$
while $|v(q,t_m)| \leq TOL$ **do**
    $t_m \leftarrow (LT+UT)/2$
    while $|u(q,t_m)| \leq TOL$ **do**
        $q \leftarrow (LB+UB)/2$
        if $u(LB,t_m) \cdot u(q,t_m) < 0$ **then**
            $UB \leftarrow q$
        end
        else
            $LB \leftarrow q$
        end
    end
    $q^* \leftarrow q$
    return $q^*, \Pi_r(q^*, tm)$
    $t_m^* \leftarrow \arg\max \Pi_r(q^*, t_m)$
    if $v(q^*, LT) \cdot v(q^*, t_m^*) < 0$ **then**
        $UT \leftarrow t_m^*$
    end
    else
        $LT \leftarrow t_m^*$
    end
end
$q^{**} \leftarrow q^*$
$t_m^{**} \leftarrow t_m^*$

retailer. The profit function of the manufacturer Π_m can be expressed as follows:

$$\Pi_m(q) = wq - c_m q$$

where w and c_m represent the wholesale price and the manufacturer's per-unit cost, respectively. For the manufacturer, the lower and upper bounds of the order quantity by the retailer can be expected. Therefore, the lower and upper bounds of the expected profit of the manufacturer are as follows:

$$\text{Lower bound} : (w - c_m) \cdot \left(\frac{t_m}{T} F^{-1} \left(\frac{p - (c_r + w)}{p} \right) + \frac{t_m}{T} (a - bp) \right)$$
$$\text{Upper bound} : (w - c_m) \cdot \left(F^{-1} \left(\frac{p - (c_r + w)}{p} \right) + a - bp + \frac{T - t_m}{T} \alpha bp \right)$$

Although the order quantity is assumed as infinite, the profit of the manufacturer occurs within the interval given as described. When the wholesale price is fixed, the profit of the manufacturer varies with t_m as determined by the retailer. Also, the determination of t_m by the retailer depends on the wholesale price. Therefore, the profit of the manufacturer depends on the optimal combination of the start time of markdown sale and the order quantity determined by the retailer in the decentralized system. A detailed analysis was conducted with numerical experiments.

2.3.4 Numerical experiments of the decentralized system

We conducted numerical experiments to analyze the decentralized system. The parameter setting for the experiments is provided in Table 2.3. We analyzed the optimal combination of the start time of the markdown sale and the order quantity determined by the retailer and how it affects the profits of the manufacturer and the supply chain system. We considered the following questions for the numerical experiments:

(i) How does the optimal order quantity vary with the change of the start time

Table 2.3: Parameter setting for the numerical experiments of a decentralized system

	p	a	b	α	w	c_r	c_m
Case 1	120	7000	50	0.4	35	20	24
Case 2	120	7000	45	0.4	35	20	24
Case 3	120	6200	45	0.4	20	15	17
Case 4	120	7500	35	0.4	20	5	8

of the markdown sale?

(ii) What is the start time of the markdown sale generating the maximum profit for the retailer?

(iii) How does the profit of the manufacturer vary?

(iv) How does the profit of the system vary?

We also solved the problem by varying the given start time of the markdown sale t_m to confirm whether or not the optimal combination proposed in this study generates the maximum profit. That is, the optimal order quantity and the expected profits of the retailer, manufacturer, and system (Π_r, Π_m, and Π_s, respectively) were estimated by fixing t_m. The results of the numerical experiments are illustrated in Figure 2.2 and detailed results are provided in Table 2.4. The optimal combination of the start time of the markdown sale and the order quantity, and the profits of the retailer, manufacturer, and system, are described in Table 2.5. The answers to questions (i) – (iv) are presented in Observations 1 - 4.

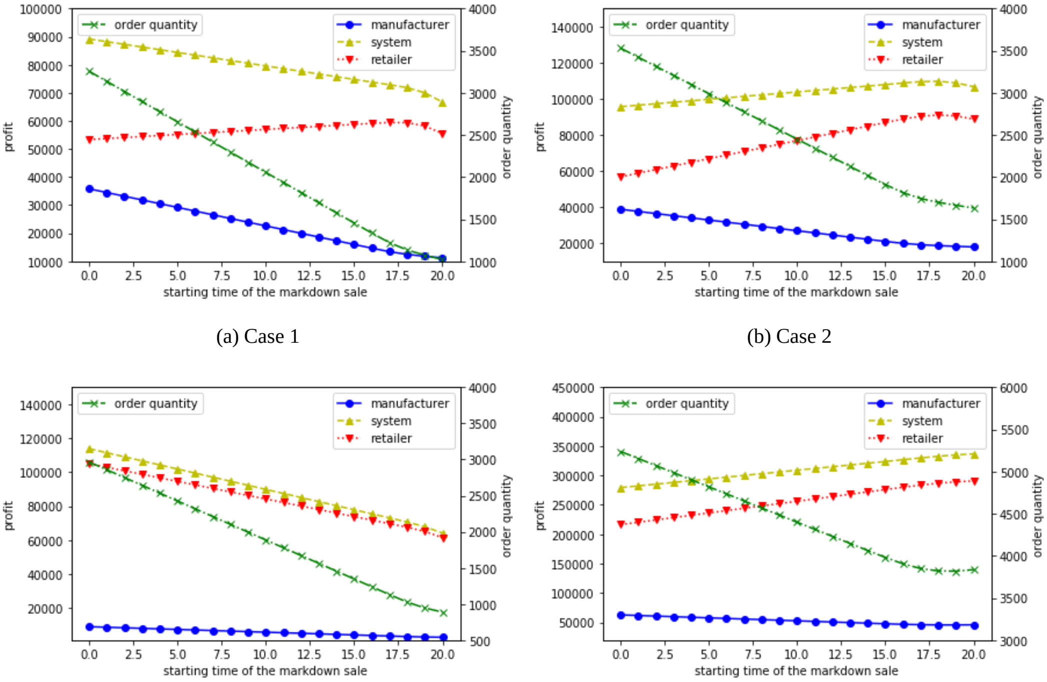
Observation 1. *As shown in Figure 2.2, when the start time of the markdown sale t_m increased, the optimal order quantity q^* of the retailer decreased.*

Because the probability distribution of the aggregated demand during the plan-

Table 2.4: Numerical experiments for a decentralized system

	Case 1				Case 2			
t	q^*	Π_r	Π_m	Π_s	q^*	Π_r	Π_m	Π_s
0	3,256	53,363	35,816	89,179	3,530	56,821	38,830	95,651
1	3,136	53,723	34,496	88,219	3,422	58,825	37,642	96,467
2	3,016	54,083	33,176	87,259	3,314	60,829	36,454	97,283
3	2,896	54,443	31,856	86,299	3,206	62,833	35,266	98,099
4	2,776	54,803	30,536	85,339	3,098	64,837	34,078	98,915
5	2,656	55,163	29,216	84,379	2,990	66,841	32,890	99,731
6	2,536	55,523	27,896	83,419	2,882	68,845	31,702	100,547
7	2,416	55,883	26,576	82,459	2,774	70,849	30,514	101,363
8	2,296	56,243	25,256	81,499	2,666	72,853	29,326	102,179
9	2,176	56,603	23,936	80,539	2,558	74,857	28,138	102,995
10	2,056	56,963	22,616	79,579	2,450	76,861	26,950	103,811
11	1,936	57,323	21,296	78,619	2,342	78,865	25,762	104,627
12	1,816	57,683	19,976	77,659	2,234	80,869	24,574	105,443
13	1,696	58,043	18,656	76,699	2,126	82,873	23,386	106,259
14	1,576	58,403	17,336	75,739	2,018	84,877	22,198	107,075
15	1,456	58,763	16,016	74,779	1,911	86,876	21,021	107,897
16	1,336	59,123	14,696	73,819	1,815	88,798	19,965	108,763
17	1,222	59,440	13,442	72,882	1,745	90,333	19,195	109,528
18	1,135	59,367	12,485	71,852	1,700	91,026	18,700	109,726
19	1,073	58,207	11,803	70,010	1,665	90,547	18,315	108,862
20	1,021	55,478	11,231	66,709	1,633	88,764	17,963	106,727

	Case 3				Case 4			
t	q^*	Π_r	Π_m	Π_s	q^*	Π_r	Π_m	Π_s
0	2,966	104,927	8,898	113,825	5,239	216,510	62,868	279,378
1	2,858	102,851	8,574	111,425	5,155	220,482	61,860	282,342
2	2,750	100,775	8,250	109,025	5,071	224,454	60,852	285,306
3	2,642	98,699	7,926	106,625	4,987	228,426	59,844	288,270
4	2,534	96,623	7,602	104,225	4,903	232,398	58,836	291,234
5	2,426	94,547	7,278	101,825	4,819	236,370	57,828	294,198
6	2,318	92,471	6,954	99,425	4,735	240,342	56,820	297,162
7	2,210	90,395	6,630	97,025	4,651	244,314	55,812	300,126
8	2,102	88,319	6,306	94,625	4,567	248,286	54,804	303,090
9	1,994	86,243	5,982	92,225	4,483	252,258	53,796	306,054
10	1,886	84,167	5,658	89,825	4,399	256,230	52,788	309,018
11	1,778	82,091	5,334	87,425	4,315	260,202	51,780	311,982
12	1,670	80,015	5,010	85,025	4,231	264,174	50,772	314,946
13	1,562	77,939	4,686	82,625	4,147	268,146	49,764	317,910
14	1,454	75,863	4,362	80,225	4,063	272,118	48,756	320,874
15	1,346	73,787	4,038	77,825	3,981	276,082	47,772	323,854
16	1,238	71,711	3,714	75,425	3,905	279,990	46,860	326,850
17	1,130	69,634	3,390	73,024	3,849	283,676	46,188	329,864
18	1,026	67,515	3,078	70,593	3,821	286,860	45,852	332,712
19	944	65,000	2,832	67,832	3,818	289,278	45,816	335,094
20	888	61,410	2,664	64,074	3,836	290,781	46,032	336,813



(a) Case 1 (b) Case 2

(c) Case 3 (d) Case 4

Figure 2.2: Numerical experiments of the decentralized system

ning horizon follows $N(a - bp + \frac{(T-t_m)}{T}\alpha bp, \sigma^2)$, when t_m becomes larger, the lower demand occurs. On the contrary, when t_m approaches 0, the total demand of the customer increases. As shown in Inequality (2.3) in Corollary 1, when t_m approaches T, the range of the lower and upper bounds of q^* is close to $F^{-1}(\frac{p-c}{p}) + a - bp$, while the range is widened when t_m approaches 0. If the sale price $(1 - \alpha) \cdot p$ is not less than the purchase cost $c_r + w$, the retailer places more order than $F^{-1}(\frac{p-c}{p}) + a - bp$ to cover the demand which is larger than the demand ξ. Consequently, the optimal order quantity q^* for the retailer tends to increase with decreasing t_m. Thus, the inverse property between the optimal order quantity of the retailer and the start

Table 2.5: Optimal combinations of the start time of the markdown sale and order quantity, and the profits of the retailer, manufacturer, and system from the numerical experiments in the decentralized system

	Case 1	Case 2	Case 3	Case 4
Start time of markdown sale	$t = 17.44$	$t = 18.12$	$t = 0.00$	$t = 20.00$
Order quantity	1,179	1,695	2,966	3,836
Profit of the retailer	59,496	91,034	104,927	290,781
Profit of the manufacturer	12,969	18,654	8,898	46,032
Profit of the system	72,465	109,688	113,825	336,813

time of the markdown sale t_m was observed.

Observation 2. *As shown in Case 3, the maximum profit of the retailer was generated when the markdown sale started at $t_m = 0$. On the contrary, in Case 4, the maximum profit of the retailer was reached when the markdown sale started at $t_m = T$. In Cases 1 and 2, the retailer would choose the optimal start time of the markdown sale t_m^* as 17.44 and 18.12, respectively.*

Corollary 2. *If the wholesale price w is set to be less than $-c_r + 2p - \alpha p - \frac{a}{b}$, it is a sufficient condition for the retailer starting the markdown sale at $t_m = 0$ where the manufacturer makes the relatively larger profit compared to the opposite case.*

Proof. Since the expected profit function of the retailer is concave with respect to t_m, when the first derivative of the function has negative value at $t_m = 0$, it also has a negative value even after $t_m = 0$. Therefore, when the wholesale price w is set to be less than $-c_r + 2p - \alpha p - \frac{a}{b}$, the retailer acquires the maximum expected profit when the markdown sale starts at $t_m = 0$ (see Appendix A.5). At this time, the optimal order quantity of the retailer is as follows:

$$\lim_{t_m \to 0+} \frac{\partial \Pi_r(q, t_m)}{\partial q} = (1-\alpha) \cdot p \cdot (1 - F(q - (a - bp) + \alpha bp)) - (c_r + w)$$

Accordingly,

$$\lim_{t_m \to 0+} q^* = F^{-1}\left(\frac{(1-\alpha) \cdot p - (c_r + w)}{(1-\alpha) \cdot p}\right) + a - bp + \alpha bp$$

□

Corollary 3. *If the following inequality is satisfied, the maximum profit of the retailer occurs when the markdown sale starts at* $t_m = T$*.*

$$(a - bp + \alpha bp)\left(\frac{p - (c_r + w)}{p}\right) - b \cdot (p - (c_r + w)) + \int_{-\infty}^{\frac{p-(c_r+w)}{p}} x f(x) dx > 0$$

Proof. The proof process of Corollary 3 is similar to that of Corollary 2. Since the expected profit function Π_r is strictly concave with respect to t_m, if the value of the first derivative of the function is positive at $t_m = T$, then the maximum profit is reached at this point (see Appendix A.6). Thus, the optimal order quantity of the retailer is as follows:

$$\lim_{t_m \to T-} \frac{\partial \Pi_r(q, t_m)}{\partial q} = p \cdot (1 - F(q - (a - bp) + \alpha bp)) - (c_r + w)$$

Accordingly,

$$\lim_{t_m \to T-} q^* = F^{-1}\left(\frac{p - (c_r + w)}{p}\right) + a - bp$$

□

Observation 3. *According to Observation 1, the optimal order quantity* q^* *by the retailer decreased as* t_m *increased. Because the profit function of the manufacturer follows* $\Pi_m = wq - c_m q$*, it is proportional to the order quantity of the retailer. Therefore, the manufacturer prefers that the retailer starts the markdown sale at* $t_m = 0$*.*

Customer demand is assumed to be controllable according to the start time of the markdown sale. From the perspective of the manufacturer, it is profitable when the retailer orders as many as possible. Therefore, the manufacturer prefers the start time of the markdown sale at $t_m = 0$ which amplifies the customer demand.

Observation 4. *The overall profit of the system also depends on the retailer's start time of the markdown sale. For example, in Case 1, which is illustrated in Figure 2.2 (a), the system profit reached the maximum value when* t_m *was determined at* $t_m = 0$*, but the retailer benefited from starting the markdown sale at another time.*

In Cases 1 and 2, the most profitable start time of the markdown sale for the retailer differed from that of the manufacturer and the overall system. Especially for Case 1, for the manufacturer or system, the maximum profit was gained when the markdown sale started at $t = 0$, but the retailer determined the start time at $t = 17.44$. Meanwhile, for Cases 3 and 4, the retailer determined the start time of markdown sale, which also generated the maximum profits of the manufacturer and system despite the decentralized system. Although the optimal start time of the markdown sale of the retailer coincided with that of the manufacturer, the supply chain was not coordinated. It is necessary to analyze the optimal combination of a start time of the markdown sale and order quantity from a system point of view. Also, an appropriate distribution of the maximum system profit is required. The supply chain coordination based on the centralized system is discussed in the next section.

2.4 Analysis of a centralized system

In this section, an optimal combination (t_m^{**}, q^{**}) from the system perspective are considered. We adopted a revenue-sharing contract rather than a buy-back contract because the newsvendor model does not consider leftover stock. The main purpose of the contract is to change the profit structure to reach the *Pareto optimum*. Based on the *Stackelberg game*, the manufacturer who is the leader determines the contract parameters, and the retailer who is the follower subsequently decides on a start time of the markdown sale and an order quantity. Under the revenue-sharing contract, the transfer payment from a retailer to a manufacturer includes a certain fraction of the retailer's revenue ℓ and the wholesale price w. The determination of the appropriate wholesale price proposed by Cachon and Lariviere [19] is modified, and the sufficient condition for this model is proposed for the revenue-sharing contract.

In the case of the decentralized system, because w is larger than c_m, the optimal order quantity from the viewpoint of the system is not placed. To establish the revenue-sharing contract, the manufacturer must provide the retailer with a wholesale price w that is less than c_m and receive a certain percentage of the revenue ℓ from the retailer.

2.4.1 Revenue-sharing contract

We suggest a revenue-sharing contract to overcome the relatively small profit of the system due to the decision from each standpoint. Under the revenue-sharing contract, the expected profit functions of the retailer and manufacturer, respectively, are as

follows:

$$\Pi_r(q,t_m) = R_1(q,t_m) + R_2(q,t_m) - (w+c_r)\cdot q - (1-\ell)\cdot(R_1(q,t_m)+R_2(q,t_m)) \tag{2.6}$$

$$\Pi_m(q,t_m) = (1-\ell)\cdot(R_1(q,t_m)+R_2(q,t_m)) + wq - c_m q \tag{2.7}$$

The total expected profit function of the supply chain system Π_s is as follows:

$$\Pi_s(q,t_m) = R_1(q,t_m) + R_2(q,t_m) - (c_r+c_m)\cdot q \tag{2.8}$$

From the system perspective, the optimal combination (t_m^{**}, q^{**}) can be obtained through the Algorithm proposed in Section 3 by replacing the wholesale price w with the manufacturer's per-unit cost c_m. Denote q_r^*, q_m^*, and q_s^* by the optimal order quantities for the retailer, manufacturer, and system, respectively. If a certain fraction of the retailer's revenue ℓ and wholesale price w satisfy Equations (2.9) and (2.10), then $q_r^* = q_m^* = q_s^*$ holds true which means the supply chain is coordinated.

$$\ell = \frac{w+c_r}{c_r+c_m} \tag{2.9}$$

$$w = -c_r + (c_r+c_m)\cdot\ell \tag{2.10}$$

Corollary 4. *If the following inequality* (2.11) *is satisfied, then the maximum profit of the system occurs when the retailer starts the markdown sale at* $t_m = 0$.

$$a - bp + \alpha bp - b\cdot(p - c_r - c_m) < 0 \tag{2.11}$$

Proof 6. *It can be easily proved by referring to the proof of the Corollary 2 by replacing the wholesale price* w *with the manufacturer's cost per-unit* c_m.

The optimal order quantity by the retailer, manufacturer, and system, respectively, are as follows:

$$\begin{aligned}
q_r^* &= F^{-1}\left(\frac{\ell \cdot (1-\alpha) \cdot p - (w + c_r)}{\ell \cdot (1-\alpha) \cdot p}\right) + a - bp + \alpha bp \\
q_m^* &= F^{-1}\left(\frac{(1-\ell) \cdot (1-\alpha) \cdot p - (c_m - w)}{(1-\ell) \cdot (1-\alpha) \cdot p}\right) + a - bp + \alpha bp \\
q_s^* &= F^{-1}\left(\frac{(1-\alpha) \cdot p - (c_r + c_m)}{(1-\alpha) \cdot p}\right) + a - bp + \alpha bp
\end{aligned}$$

When Equations (2.9) and (2.10) are satisfied, then $q_r^* = q_m^* = q_s^*$ holds true and the supply chain is coordinated.

Corollary 5. *If the following inequality* (2.12) *is satisfied, then the maximum profit of the system occurs when the retailer starts the markdown sale at* $t_m = T$.

$$(a - bp + \alpha bp)\left(\frac{p - (c_r + w)}{p}\right) - b \cdot (p - c_r - c_m) + \int_{-\infty}^{\frac{p-(c_r+c_m)}{p}} x f(x) dx > 0 \qquad (2.12)$$

Proof 7. *The proof of Corollary 5 can be easily completed by replacing the wholesale price with the manufacturer's cost per-unit in the proof of Corollary 3.*

In this case, the optimal order quantity by the retailer, manufacturer, and system, respectively, are as follows:

$$\begin{aligned}
q_r^* &= F^{-1}\left(\frac{\ell \cdot p - (w + c_r)}{\ell \cdot p}\right) + a - bp \\
q_m^* &= F^{-1}\left(\frac{(1-\ell) \cdot p - (c_m - w)}{(1-\ell) \cdot p}\right) + a - bp \\
q_s^* &= F^{-1}\left(\frac{(p - (c_r + c_m)}{p}\right) + a - bp
\end{aligned}$$

When Equations (2.9) and (2.10) hold true, the supply chain is coordinated with

Table 2.6: Optimal combinations of the start time of the markdown sale and order quantity, and the profits of the retailer, manufacturer, and system from the numerical experiments in the centralized system

	Case 1 ($\ell = 0.75$)	Case 2 ($\ell = 0.83$)	Case 3 ($\ell = 0.92$)	Case 4 ($\ell = 0.86$)
Start time of markdown sale	$t = 0.00$	$t = 17.62$	$t = 0.00$	$t = 20.00$
Order quantity	3,269	1,809	2,982	4,115
Profit of the retailer	67,260	91,523	104,946	290,998
Profit of the manufacturer	22,420	18,746	8,903	47,372
Profit of the system	89,680	110,269	113,849	338,370

$q_r^* = q_m^* = q_s^*$. Otherwise, the optimal combination of the start time of the markdown sale and order quantity can be obtained by the Algorithm with the use of c_m instead of w. Under the coordination, the optimal combination (t_m^{**}, q^{**}) from the system is also the optimal combination for the retailer and the manufacturer. By inserting the optimal combination (t_m^{**}, q^{**}) into the profit functions (2.6), (2.7), and (2.8), the maximum expected profits of the retailer, manufacturer, and system, respectively, are obtained.

2.4.2 Numerical experiments of the centralized system

Numerical experiments were conducted to characterize the centralized system. The parameter setting from Table 2.3 was also used for the experiments, except for the wholesale price w and a certain fraction of the retailer's revenue ℓ. A summary of the numerical experiment is provided in Table 2.6.

As can be seen from Table 2.6, all expected profits were higher than those shown in Table 2.5. The optimal order quantities by the retailer, manufacturer, and system were equal, showing that the supply chain was coordinated. Although setting w as less than c_m led to the revenue of the manufacturer as a negative value, the profit of the manufacturer was higher than the decentralized system. Notably, for Cases 1, 2, and 4, the optimal order quantities increased through coordination, but order quantity decreased in Case 3. In this case, the wholesale price w was set to be relatively small, according to Corollary 2. For the retailer, the total purchasing cost was reduced, which resulted in placing a larger order quantity. Table 2.6 shows that the optimal start time of the markdown sale was consistent with the decentralized system in Cases 3 and 4. In Cases 1 and 2, however, the optimal start time of the markdown sale changed with the coordination. When the start time of markdown sale is considered, supply chain coordination is required to match the optimal start time of markdown sale as well as the optimal order quantity.

2.5 Summary

As the inventory management for perishable items becomes critical to a company, it is necessary to adjust the customer demand by taking the start time of the markdown sale into consideration at the end of the selling season. In preparation for the end of the selling season, the retailer should consider not only the order quantity but also the start time of the markdown sale. However, it can be disadvantageous to the manufacturer because the profit depends on the decision of the retailer. That is, the decision in each perspective cannot reach the system's maximum profit. The maximum profit from the system's perspective can be obtained by utilizing a revenue-sharing contract based on the appropriate fraction of the retailer's revenue and wholesale price.

The proposed newsvendor model and revenue-sharing contract could be useful to the inventory manager when the end of selling season approaches. At the end of the selling season, the inventory manager will place the last order and determine the start time of the markdown sale. By utilizing the output of this study, the whole supply chain system, including the retailer and manufacturer, could obtain the maximum profit. This study can also be useful when the perishable item is discounted within a short period. In the case of groceries, which are only sold on the same day in a market, it can be easily observed that they are sold with the sale price in the evening. Another example could be a fast-fashion product, which is produced only once, and the remaining inventories are run out with the markdown sale.

2.5.1 Managerial insights

Research on the newsvendor model and supply chain coordination has been widely conducted. Also, variations of the model with regard to pricing have been widely developed. To the best of our knowledge, however, research considering the start time of a markdown sale is scarce. Our study produced the following managerial insights:

(i) When a retailer sets the start time of the markdown sale individually, it can result in being quite disadvantageous to the manufacturer. The profit of the manufacturer depends on the order quantity of the retailer when the wholesale price is fixed. Therefore, the manufacturer prefers that the retailer starts the markdown sale as soon as possible to amplify customer demand. However, the retailer determines the optimal order quantity depending on the given external factors; that is, it is difficult to match each preference of a start time of the markdown sale in the decentralized system.

(ii) When the manufacturer sets the wholesale price appropriately, the preferred start time of the markdown sale from the retailer and manufacturer can coincide. In this case, the profit difference between the decentralized and centralized system is relatively small, but the supply chain is not coordinated.

(iii) In general, a supply chain coordination based on the newsvendor model means that the optimal order quantity from the system point of view equals that of the retailer and manufacturer while the system profit is maximized. However, when the concept of determining the optimal start time of the markdown sale is introduced, not only matching the order quantity but also the start time must be considered. In other words, the supply chain coordination is achieved when the optimal combination (t_m^{**}, q^{**}) is realized.

Chapter 3

Robust Multiperiod Inventory Model with a New Type of Buy One Get One Promotion: "My Own Refrigerator"

3.1 Introduction and literature review

Among the various types of sales promotions, the most frequently encountered in daily life is a *price reduction*. For instance, airline companies and hotels reduce the price to promote the sale of remaining seats and rooms, respectively. In the case of a supermarket, the company reduces the prices of perishable foods each day as closing time approaches. A similar tactic over a longer time scale can be observed in the fashion industry, where a company stimulates customer demand through markdowns (clearance sales) at the end of the selling season. Another common promotion is a *buy one get one free* (BOGO) promotion. This promotion looks similar to a price reduction but can be more effective at attracting customers. According to Shampanier et al. [89], customers generally overvalue the benefit of *free* compared to a discounted price. Furthermore, it can undoubtedly reduce stocks further than a price reduction, under the assumption that the same number of customers arrive to purchase.

Under a BOGO promotion, however, customers who want to buy a relatively small quantity of products could be provided with more products than necessary. In the case of customers visiting a convenience store, they might be limited by the weight of the product, the capacity of their refrigerator at home, or the short shelf life of a perishable product. To relieve these limitations while retaining the advantages of the BOGO promotion, *GS Retail*, one of the largest retail companies in Korea, which operates more than 8,000 *GS*25 convenience stores, launched the "*My Own Refrigerator*" (MOR) mobile application. Customers who use the MOR application can delay taking the *second product* (*freebie*), put it in their virtual storage, and pick it up another day. This option eliminates the concerns regarding heavy loads, storage capacity, and short product shelf life. As a result, more than ten million users have downloaded the MOR application since GS Retail launched it in March 2011.

From the retailer's standpoint, it is possible to amplify customer demand through BOGO promotions with the MOR application. Accordingly, high revenue and customer satisfaction can be expected. However, the retailer still incurs the inventory holding cost even after the products have actually been sold because customers retain second products in their virtual storage, which corresponds to the actual shelf of the retailer. Thus, these products not only incur holding costs but also occupy capacity. Even if the product remains in the store's inventory, it is a product that has already been sold to the customer and can no longer generate profit. Most of all, there is high uncertainty as to when customers will pick up their second products. Therefore, the retailer must order products taking into account the quantity of the products on the shelf that have already been sold. This suggests that a new method for inventory control is required.

The demand of customers who arrive at the store to buy the promoted products can be estimated based on accumulated historical data over a long period. Concerning the second product, relatively more uncertainty exists as to whether the customers will pick up the product that day or not, or when the customers will revisit the store. Furthermore, if customers who have already purchased products through the MOR application face stockouts when they revisit the store to pick up their second products, brand loyalty could drop sharply. BOGO promotions through MOR can increase customer demand and generate high revenue, but they are also accompanied by a significant potential penalty to brand image. To deal with the uncertainty of revisiting customer demand, we considered a distributionally robust optimization approach to the multiperiod inventory model as a countermeasure against the worst-case distribution. Under the distributionally robust optimization approach, a family of distributions, including the true distribution, is considered. Only partial information from the descriptive statistics on the data is required to describe the uncertain ambiguity demand. In this study, we considered the moment-based partial information, such as means, covariance, and supports of uncertain revisiting rates.

Robust optimization has been actively studied in the context of decision-making under uncertain data. In the robust optimization scheme, input data have been regarded as an uncertain value belonging within a particular range rather than as a nominal value. Robust optimization seeks the optimal solution under the worst-case scenario that guarantees the feasibility of all possible realizations of uncertainty from the input data. For the inventory manager, it is very important to build a flexible and robust model that allows the company to respond to customer demand with a high service level ([22]). Soyster [100] first proposed the robust optimization model with a box shape for the uncertainty set. Since then, research has progressed on the structure of the set, providing a less conservative solution while preserving the feasibility guarantee and tractability. Ben-Tal and Nemirovski [10, 12] developed the robust optimization model under the uncertainty set in the form of an ellipsoid shape, which provides a less conservative solution. The polyhedral uncertainty set was then developed by Bertsimas and Sim [14]. The box, ellipsoid, and polyhedral shapes have been considered as the standard forms of the uncertainty set in the robust optimization context. Various applications have been studied based on the fruitful development of robust optimization theory (see [13, 35, 38]).

Most of the abovementioned studies are based on decision-making in a static situation. A decision made before the realization of all uncertain data is commonly referred to as a *here and now* decision. In contrast, a *wait and see* decision can be applied more naturally in a multistage decision process, based on the partial realization of uncertain data. In the wait and see decision, decision variables are separated by *adjustable variables* and *nonadjustable variables*. Decision variables that are determined after the realization of uncertain data are called adjustable variables. In contrast, decision variables that are determined before the realization of uncertain data are called nonadjustable variables. By partitioning decision variables in this manner, an *adjustable robust optimization* was established ([9]). Although this approach could permit the flexibility of modeling, the adjustable robust counterpart is generally an NP-hard problem, which is computationally intractable. Accordingly, Ben-Tal et al. [9] derived the computationally tractable formulation by restricting the decision variables to the parameterized functions, which is linearly (affinely) dependent on uncertain factors.

Meanwhile, the distributionally robust optimization generalizes the stochastic optimization and robust optimization. By considering the ambiguity set, which contains the true distribution, but the distribution is not known, the optimal solution is provided based on the worst-case distribution. If only the true distribution is considered, it becomes the stochastic optimization problem. On the contrary, if all distributions under the given support set are considered, it becomes the robust optimization ([80]). In other words, the distributionally robust optimization generalizes the stochastic optimization and robust optimization. Accordingly, the distributionally robust optimization provides less conservative solution than the robust optimization. Also, it could retain the tractability from the primal optimization problem.

We investigated the previous research, which considered the (distributionally) robust multiperiod inventory problems in detail. Bertsimas and Thiele [15] applied a robust optimization to the multiperiod inventory model based on a polyhedral uncertainty set. Although the model was developed based on a here and now decision in the multiperiod setting, the robust counterpart was developed as a tractable linear program. They adopted a budget of uncertainty, proposed by Bertsimas and Sim [14], by forcing the independence of the uncertain data over the period. Ben-Tal et al. [8] adapted the adjustable robust optimization approach to the retailer-supplier flexible commitment contract, which reduces the bullwhip effect by imposing a penalty on a violation of the promised order quantity in advance between the retailer and supplier. By developing the problem as an affinely adjustable robust counterpart with a min-max criterion, they efficiently solved the problem against the worst-case scenario. Subsequently, See and Sim [88] solved the multiperiod inventory problem whose objective function is presented as the expectation under stochastic demand. They considered stochastic demand as a factor-based demand model that is an affine function of the uncertainty factors. In detail, they utilized the distributionally robust bound presented by Chen and Sim [26] and linear decision rule to derive the tractable formulation, which features a second-order cone problem. Meanwhile, Goh and Sim [37] developed ROME, a software program for solving the robust optimization problems. They also presented three problems: inventory, project crashing, and portfolio selection problems. For the inventory problem, which is the most relevant to this study, they modeled the problem with a constraint that requires satisfaction of the fill rate rather than imposing a penalty cost on stockout inventories. They applied a distributionally robust optimization approach to the fill-rate constraint for all candidates of the distributions. Ang et al. [3] adopted the distributionally robust optimization approach to the storage assignment problem. They considered a factor-based demand model and solved the problem with the linear decision rule. Another application in multiperiod inventory control is the empty container repositioning problem. Tsang and Mak [106] and Lee and Moon [60] adapted the linear decision rule and distributionally robust optimization approach to the empty container repositioning problem. For additional robust optimization applications, we refer the reader to the review paper examined by Yanikoglu et al. [117].

As can be seen from the abovementioned studies, the structure of the inventory model depends on the modeling method of the stochastic demand. If the demand for purchasing the BOGO product is restricted to a deterministic value and the demand of revisiting customers who collect the second product is subject to uncertainty, the latter can be developed as an affine function of uncertainty factors. Accordingly, it

Table 3.1: Comparisons of this research and previous relevant studies

Authors (year)	**Uncertain demand**	**Decision-making policy**	**Mean and support of uncertainty factor**	**Constraint of uncertainty factor**
‘ et al. (2004) [9]	Box	LDR[1]	N/A[2]	N/A
Ben-Tal et al. (2005) [8]	Box & Ellipsoid	LDR	N/A	N/A
Bertsimas and Thiele (2006) [15]	Polyhedron	Base stock policy	N/A	N/A
Ben-Tal et al. (2009) [7]	Box	LDR	N/A	N/A
Wei et al. (2011) [111]	Polyhedron	LDR	N/A	N/A
See and Sim (2010) [88]	Factor-based demand model	LDR	Zero mean and bounded support $[-\underline{z}, \overline{z}]^*$	Unconstrained
Goh and Sim (2011) [37]	Factor-based demand model	LDR	Non-zero mean and positive bounded support $[0, \overline{z}]$	Unconstrained
Ang et al. (2012) [3]	Factor-based demand model	LDR	Non-zero mean and bounded support $[-\underline{z}, \overline{z}]$	Unconstrained
Tsang and Mak (2015) [106]	Factor-based demand model	LDR	Zero mean and bounded support $[-\underline{z}, \overline{z}]$	Unconstrained
Lee and Moon (2019) [60]	Factor-based demand model	LDR	Zero mean and bounded support $[-\underline{z}, \overline{z}]$	Unconstrained
This research	Factor-based demand model	LDR	Non-zero mean and positive bounded support $[0, \overline{z}]$	Constrained

LDR[1] and N/A[2] indicate the linear decision rule and not applicable, respectively.
*For the bounded support, $\underline{z}$ and $\overline{z}$ are positive.

has the same property in a factor-based demand model. In this study, the sum of the uncertainty factors in a particular interval is constrained to less than or equal to 1. This is different from previous studies, which assumed the uncertainty factors as unconstrained random variables. Also, uncertainty factors considered in this study are not zero-mean random variables. To distinguish the characteristics of this study from the previous research, we summarize the relevant literature in Table 3.1.

From the perspective of the application and modeling, the main contributions of this study are as follows:

- We applied the distributionally robust optimization approach to deal with uncertainty in operating the real-world mobile application. Through various experiments, managerial insights were identified that would be helpful to the retailer.

- Previous studies considering the factor-based demand model have mostly assumed zero-mean and unconstrained random variables. In this study, the non zero-mean random variable is considered, and the sum of uncertainty factors over the periods is constrained. We developed a robust counterpart that incorporates these features and described the process in detail.

The remainder of the chapter is organized as follows: We introduce the problem description of the inventory model for My Own Refrigerator (IMMOR) in Section 3.2. Section 3.3 deals with the mathematical formulation of the IMMOR. In Section 3.4, we present computational experiments and analyses. In Section 3.5, we summarize the findings of this research.

3.2 Problem description

We consider a single-item multiperiod inventory model based on the discrete-time planning horizon $t \in \{1, \ldots, T\}$. In this chapter, we will use the term *purchasing demand* as the demand of the customer who visits the store to buy the BOGO product through the MOR application. For the demand of the customer who has already made a payment and drops by the store to pick up a second product, we will use the term *revisiting demand*. Each customer can take both products at once or take one and revisit in the future to pick up the second product. These two types of demand for each period are illustrated in Figure 3.1. In practice, purchasing demand is more predictable than revisiting demand because historical data for the former have accumulated over a long period. Revisiting demand is less predictable since there is relatively little accumulated data. We consider purchasing demand as a deterministic demand and revisiting demand as a stochastic demand. For this study, we made the following assumptions:

Assumption 1. *Customers purchase one package of a promotion product (a set of two products) and take either one or both products at the time of purchase.*

Assumption 2. *It is unknown when the customers will revisit the store to pick up the second product, but they will revisit before the expiry date from the purchasing date* $[t, t + \tau)$.

Assumption 3. *Purchasing demand is the deterministic demand, and revisiting demand is the stochastic demand.*

Assumption 4. *The revisiting rate in the last period* $(t = T - 1)$ *has the value of 1. In other words, customers who buy the BOGO product in the last period take two*

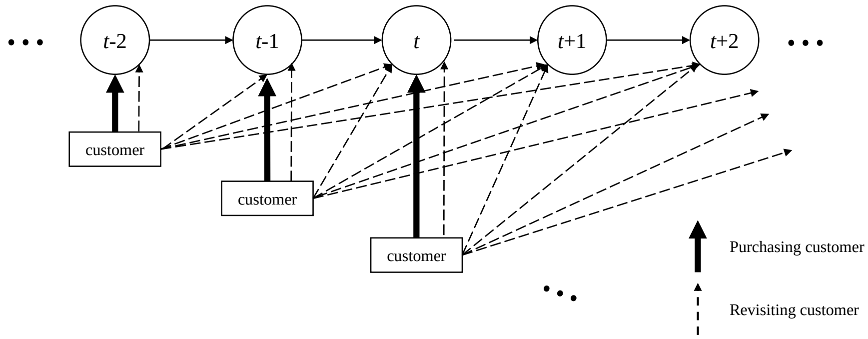


Figure 3.1: Two types of demands for each period

products because they know that they cannot take the second product in the future.

Assumption 5. *The BOGO promotion through the MOR application is valid for a given planning horizon. That is, we assume that it is available from $t = 1$ and ends without salvage value after $t = T$.*

The assumptions in this study are made based on the operation of MOR in practice. For more information about the application, we refer readers to the *App Store*, *Google Play*, or the website of GS25 (`http://gs25.gsretail.com`). Throughout this chapter, we define $\mathfrak{T} \triangleq \{1, \ldots, T\}$ and $\mathfrak{T}^{-} \triangleq \{1, \ldots, T-1\}$ for brevity in expressing the planning horizon.

3.2.1 Demand modeling

Let d_t and $\tilde{\xi}_t$ denote the deterministic purchasing demand and stochastic revisiting demand, respectively. Denote by $\tilde{\boldsymbol{\rho}}^t \triangleq \left(\tilde{\rho}^t_t, \tilde{\rho}^t_{t-1}, \ldots, \tilde{\rho}^t_{t-\tau+1}\right)$ a vector of the *revisiting rate*, where τ is an expiry date from the purchase date. Each revisiting rate means the probability of taking both products at period t, the probability of revisiting to collect the second product at period t from the period $t-1$, ... , the probability of revisiting to collect the second product at period t from the period $t-\tau+1$. A set of vectors can be represented by a matrix $\tilde{\mathbb{A}}$, which is illustrated in Figure 3.2. We assume that d_t occurs in period $t \in \mathfrak{T}^-$ and it is scattered by a vector of uncertainty factor $\tilde{\boldsymbol{\rho}}^t$. Since each uncertainty factor follows the probability distribution, it has a value between 0 and 1. Also, the sum of the probabilities from t to $t+\tau-1$ is less than or equal to 1. The sum of these probabilities can be 1, but there is no guarantee that all customers will pick up their second products. Therefore, we set the sum to less than or equal to 1. Revisiting demand $\tilde{\xi}_t$ can be modeled as an affine function of uncertainty factors $\tilde{\boldsymbol{\rho}}^t$ as follows:

$$\tilde{\xi}_t(\tilde{\boldsymbol{\rho}}) \triangleq \textstyle\sum_{i=\max(1,t-\tau+1)}^{t} d_i \tilde{\rho}_i^t \tag{3.1}$$

$$\text{where} \begin{cases} \sum_{i=\max(1,t-\tau+1)}^{t} \tilde{\rho}_i^t \leq 1 & \\ 0 \leq \tilde{\rho}_i^t \leq 1 & i \in \{\max(1, t-\tau+1), \ldots, T-1\} \end{cases} \tag{3.2}$$

Each revisiting rate $\tilde{\boldsymbol{\rho}}^t$ in period $t \in \mathfrak{T}^-$ is constrained by (3.2). The parameters related to the demand are summarized as follows:

d_t Deterministic purchasing demand in period t

$\mathbf{d_t}$ A vector of purchasing demands from period 1 to $T-1$, $\mathbf{d_t} \triangleq (d_1, d_2, \ldots, d_{T-1})$

$\tilde{\rho}_i^t$ Revisiting rate from period i to t which is an unknown coefficient

$\tilde{\boldsymbol{\rho}}^t$ A vector of revisiting rates from period t to $t-\tau+1$

$\tilde{\xi}_t$ Stochastic demand, which is the aggregated demand of revisiting demands at period t

$\tilde{\boldsymbol{\xi}}_t$ A vector of stochastic demands from period 1 to $T-1$, that is, $\tilde{\boldsymbol{\xi}}_t \triangleq \mathbf{d_t}'\tilde{\mathbb{A}}$ and $\tilde{\boldsymbol{\xi}}_t = (\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_{T-1})$

$$
\widetilde{\mathbb{A}} =
\begin{array}{cc}
 & \overbrace{\underbrace{\hphantom{\rho_1^1 \; \rho_1^2 \; \cdots \; \rho_1^{\tau-1} \; \rho_1^{\tau}}}_{\tau} \hphantom{\; 0 \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; 0}}^{T-1} \\
T-1 \left\{ \right. &
\left[
\begin{array}{ccccccccccccccccc}
\rho_1^1 & \rho_1^2 & \cdots & \rho_1^{\tau-1} & \rho_1^{\tau} & 0 & \cdots & & & & & & & & & \cdots & 0 \\
0 & \rho_2^2 & \rho_2^3 & \cdots & \rho_2^{\tau} & \rho_2^{\tau+1} & 0 & \cdots & & & & & & & & & \vdots \\
\vdots & 0 & \ddots & & & & \vdots & \ddots & & & & & & & & & \\
 & \vdots & & \ddots & & & \vdots & & \ddots & & & & & & & & \\
 & & & & \ddots & & & & & \ddots & & & & & & & \\
 & & & & & \ddots & & & & & \ddots & & & & & & \\
 & & & & & & & \rho_{T-\tau+1}^{T-\tau+1} & \rho_{T-\tau+1}^{T-\tau+2} & \cdots & & \rho_{T-\tau+1}^{T-1} & \rho_{T-\tau+1}^{T} & & & & \\
 & & & & & & & & \ddots & & & & & \ddots & & & \\
 & & & & & & & & & & & & & & \rho_{T-3}^{T-3} & \rho_{T-3}^{T-2} & \rho_{T-3}^{T-1} \\
\vdots & & & & & & & & & & & & & \cdots & 0 & \rho_{T-2}^{T-2} & \rho_{T-2}^{T-1} \\
0 & \cdots & & & & & & & & & & & & & \cdots & 0 & \rho_{T-1}^{T-1}
\end{array}
\right]
\end{array}
$$

Figure 3.2: Matrix of uncertainty factors representing the revisiting rates

3.2.2 Sequences of the ordering decision

The inventory manager observes the inventory level at the beginning of each period and determines the order quantity to respond to future demand. We assume backlog for understocked inventory. Accordingly, the balance equation (flow conservation) among inventory level, order quantity, and demand is satisfied in each period. Also, we assume that order quantity cannot exceed an upper limit for each period. The main objective is to identify a decision that minimizes the total cost of the planning horizon $t \in \mathfrak{T}$ while satisfying the balance equation and capacity of order quantity. Without loss of generality, we assume the lead time of replenishment as 0. That is, if the product is ordered at the beginning of the period t, it is replenished in the inventory just prior to the beginning of the period $t+1$. In a given planning horizon, the order can be placed until $t = T-1$, and the salvage value of the inventory level

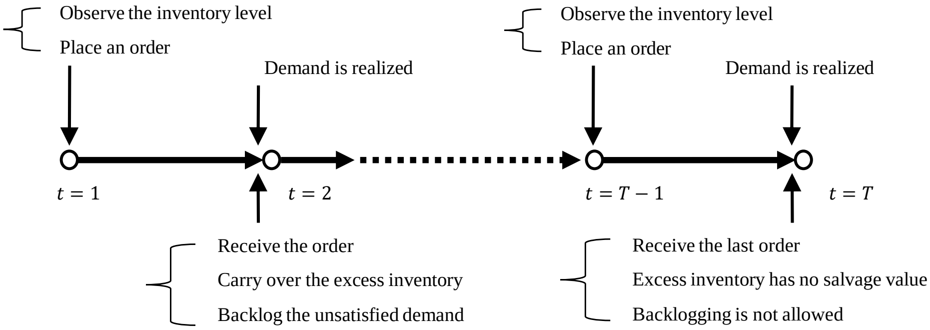


Figure 3.3: Sequences of decision making in the planning horizon

is 0 from the period T. The sequence of decision-making in the planning horizon is illustrated in Figure 3.3.

3.3 Mathematical formulation of the IMMOR

We considered two types of decision variables to respond to purchasing demand and revisiting demand. The decision variables x_t and y_t represent the order quantity to satisfy the demand d_t and $\tilde{\xi}_t$, respectively. The inventory manager determines the order quantities x_t and y_t from period $t = 1$ to period $t = T - 1$. For each order, a unit purchasing cost c_t occurs for x_t and y_t because they are the order quantities for the same item. We assume that backlogging for each inventory level is allowed. Accordingly, the inventory levels for each demand are represented by u_t and v_t, respectively, where $t \in \mathfrak{T}$. If there is overstock (understock) at the end of each period, a unit inventory holding (backlog) cost occurs for each product. It is assumed that the same unit inventory holding cost h_t occurs for the positive values of u_t and v_t. For the negative values of u_t and v_t, different unit backlog costs, b_t and p_t, respectively, are assumed. We consider two types of unit backlog cost ($b_t \ll p_t$) because the understocking revisiting demand is assumed to affect the brand image, which incurs a significant opportunity cost. Considering the capacity of the order quantity, the sum of x_t and y_t is restricted to an upper limit K_t in each period. Balance equations for purchasing demand and revisiting demand are illustrated in Figures 3.4 and 3.5, respectively.

If the order quantity and inventory level are managed by one type of decision variable, it is difficult to figure out which demand is not satisfied. In addition, a preferential response to revisiting demand is required. By partitioning the decision variables for order quantity and inventory level to the two types of decision variables (x_t, y_t, and u_t, v_t), and limiting the total order quantity by assigning an enormous backlog cost to the stockout of revisiting demand, the abovementioned issues can

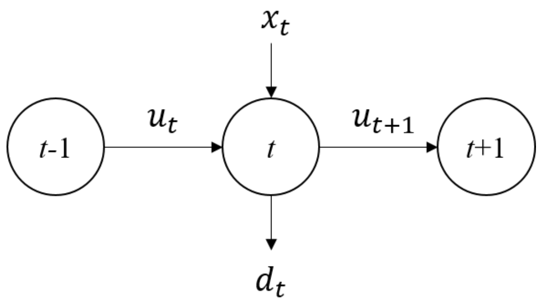


Figure 3.4: Balance equation related to purchasing demand

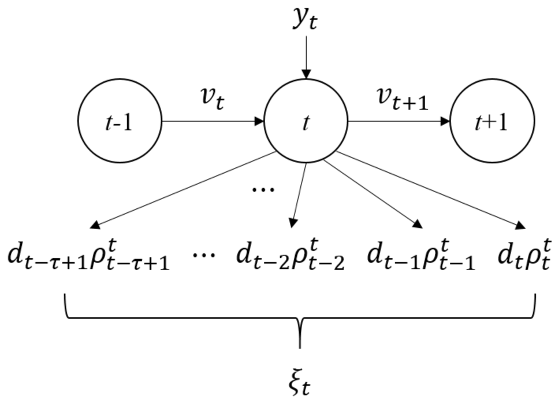


Figure 3.5: Balance equation related to revisiting demand

be handled. The inventory manager will replace the order by giving priority to revisiting demand. Meanwhile, a fixed cost for replenishment can be considered, and the decision variables can be regarded as integer values. Various types of costs, such as remanufacturing, carbon emissions, defective items, or supplier selection, could be considered to make an inventory model more realistic ([23, 24, 85, 86, 107]). In this study, however, we formulated a mathematical model as a linear program including purchasing, inventory holding, and backlog costs to retain tractability in the robust optimization approach.

3.3.1 Mathematical formulation of the IMMOR under the deterministic demand

In this section, we present a mathematical formulation based on a linear program under deterministic demand. Before making the order decision for the entire period, the inventory manager regards the uncertainty factor as a deterministic value. Consequently, revisiting demand is also considered as a deterministic value. The mathematical formulation under deterministic demand can be developed as follows:

$$
\begin{aligned}
\min \quad & \sum_{t \in \mathfrak{T}^-} \left[c_t(x_t + y_t) + h_t(u_{t+1})^+ + h_t(v_{t+1})^+ + b_t(u_{t+1})^- + p_t(v_{t+1})^- \right] & \\
\text{s.t.} \quad & u_{t+1} = u_t + x_t - d_t & t \in \mathfrak{T}^-; \\
& v_{t+1} = v_t + y_t - \xi_t & t \in \mathfrak{T}^-; \\
& x_t + y_t \le K_t & t \in \mathfrak{T}^-; \\
& x_t, y_t \ge 0 & t \in \mathfrak{T}^-;
\end{aligned} \tag{3.3}
$$

3.3.2 Mathematical formulation of the IMMOR under the stochastic demand

If revisiting demand is regarded as a random variable as shown in (3.1), a multistage stochastic optimization model can be considered. In this case, the objective function is expressed as an expectation form $\mathbb{E}(\cdot)$ and all decision variables are affected by uncertainty factors. Accordingly, the multistage stochastic optimization model can be developed as follows:

$$
\begin{aligned}
\min \quad & \mathbb{E}[\sum_{t\in\mathfrak{T}^-}(c_t(x_t(\tilde{\boldsymbol{\rho}}^{t-1})+y_t(\tilde{\boldsymbol{\rho}}^{t-1}))+h_t(u_{t+1}(\tilde{\boldsymbol{\rho}}^t))^+ + h_t(v_{t+1}(\tilde{\boldsymbol{\rho}}^t))^+ \\
& + b_t(u_{t+1}(\tilde{\boldsymbol{\rho}}^t))^- + p_t(v_{t+1}(\tilde{\boldsymbol{\rho}}^t))^-)] \\
\text{s.t.} \quad & v_{t+1}(\tilde{\boldsymbol{\rho}}^t) = v_t(\tilde{\boldsymbol{\rho}}^{t-1}) + y_t(\tilde{\boldsymbol{\rho}}^{t-1}) - d_t && t\in\mathfrak{T}^-; \\
& v_{t+1}(\tilde{\boldsymbol{\rho}}^t) = v_t(\tilde{\boldsymbol{\rho}}^{t-1}) + y_t(\tilde{\boldsymbol{\rho}}^{t-1}) - \tilde{\xi}_t(\tilde{\boldsymbol{\rho}}^t) && t\in\mathfrak{T}^-; \\
& x_t(\tilde{\boldsymbol{\rho}}^{t-1}) + y_t(\tilde{\boldsymbol{\rho}}^{t-1}) \le K_t && t\in\mathfrak{T}^-; \\
& x_t(\tilde{\boldsymbol{\rho}}^{t-1}), y_t(\tilde{\boldsymbol{\rho}}^{t-1}) \ge 0 && t\in\mathfrak{T}^-;
\end{aligned}
\tag{3.4}
$$

In practice, it is difficult to obtain full information on random demand, such as what distribution it follows. Even if the distribution is estimated, evaluating the multistage expectation is intractable due to an exponential growth of the number of scenarios according to the number of uncertain parameters. Even a two-stage stochastic model is generally an NP-hard problem unless a relatively complete recourse is guaranteed ([90, 91, 92]). In general, an *intractable problem* refers to the problem when there is no a solvable algorithm within a polynomial time. To reduce the complexity of the multistage stochastic optimization problem, the *sample average approximation* method or *stage-wise independent* process could be considered. However, we developed the demand model as constrained over the period, which cannot be assumed as independent or stage-wise independent. Therefore, instead of directly minimizing the expectation of the objective function based on the stochastic optimization approach, we focused on minimizing the approximated upper bound of the objective function. By using the linear decision rule and distributionally robust bound, we derived the tractable formulation from the multistage stochastic optimization model.

If the revisiting demand is considered as an uncertain parameter, which belongs to the specific uncertain set with the given support, and the objective function is minimized under the worst-case scenario for all possible realizations, this problem becomes a robust optimization problem. More generally, if the ambiguity demand, with the moment-based information, such as mean, covariance, and support, is considered, and the objective function is minimized under the worst-case distribution or worst-case expected cost, this problem becomes a distributionally robust optimization problem. If all distributions under the given support set are considered, the distributionally robust optimization problem reduces to the robust optimization. In other words, the distributionally robust optimization could provide a less conservative solution because of the flexibility of modeling. Accordingly, we adopted the concept of the distributionally robust optimization approach to describe the uncertain revisiting demand.

3.3.3 Distributionally robust optimization approach for the IMMOR

Instead of assuming the complete knowledge about the probability distribution, an ambiguity set, which is set of the candidate distributions, is considered. To describe the ambiguity in demand distributions, moment-based partial information, including means, covariance, and supports, is assumed to be estimated from the descriptive statistics on the data. The most common form of the factor-based demand model in the distributionally robust optimization context is as follows:

$$d_t(\tilde{\mathbf{z}}_{t-1}) \triangleq d_t^0 + \sum_{k=1}^{N} d_t^k \tilde{z}_k \tag{3.5}$$

where $1 \leq N_1 \leq N_2 \leq \cdots \leq N_{T-1} = N$ and the predefined uncertainty factors, $\tilde{\mathbf{z}}_k$, are unfolded until $k = 1, \cdots, N_t$.

Recall that stochastic demand (3.1) is also an affine function of uncertainty factors $\tilde{\boldsymbol{\rho}}$. It indicates that the demand model (3.1) can be interpreted as a special case of the factor-based demand model (3.5). Accordingly, we considered the demand represented in (3.1) as the factor-based demand model. Although the factor-based demand model could be induced from the stochastic process, such as an *auto − regressive moving average* model, we used the demand model developed as (3.1) for the following reason: The distinctive feature of the revisiting demand model is that revisiting rates are constrained over the period. To incorporate this constrained feature in the demand model, the revisiting demand was developed, as shown in (3.1).

Linear decision rule

To solve the inventory problem under the factor-based demand model, we adopted the *linear decision rule* (for the sake of brevity, we will hereafter use the abbreviation "LDR"). By restricting decision variables as affinely dependent on the uncertainty factors, the inventory manager can delay the decision by observing the realization of part of the uncertainty factors. Let ${x_t}^{\mathrm{LDR}}(\tilde{\boldsymbol{\rho}}^{t-1})$ and ${y_t}^{\mathrm{LDR}}(\tilde{\boldsymbol{\rho}}^{t-1})$ denote the order decisions based on the LDR as follows :

$$
\begin{aligned}
{x_t}^{\mathrm{LDR}}(\tilde{\boldsymbol{\rho}}^{t-1}) &= x_t^0 + \mathbf{x_t}'\tilde{\boldsymbol{\rho}}^{t-1} \\
{y_t}^{\mathrm{LDR}}(\tilde{\boldsymbol{\rho}}^{t-1}) &= y_t^0 + \mathbf{y_t}'\tilde{\boldsymbol{\rho}}^{t-1}
\end{aligned}
$$

Since the decision is based on the realized uncertainty factors, which is referred to as the *non-anticipative* property, we restricted the uncertainty factors that are unavailable in period t. It can be incorporated by summing $x_t^{i,j}\tilde{\rho}_i^j$ and $y_t^{i,j}\tilde{\rho}_i^j$ until $(i,j) \in \{(i,j) \mid i : i \leq j, j : j \leq t-1\}$. For brevity in representing the indices, let $\mathcal{N}_j \triangleq \{i \mid i \leq j\}$ and $\mathcal{M}_t \triangleq \{j \mid j \leq t-1\}$. Based on the LDR, the order quantity for purchasing demand in each period is expressed as follows:

$$x_t{}^{\mathrm{LDR}}(\tilde{\boldsymbol{\rho}}^{t-1}) = x_t^0 + \sum_{j\in\mathcal{M}_t}\sum_{i\in\mathcal{N}_j} x_t^{i,j}\tilde{\rho}_i^j \tag{3.6}$$

The decision based on the LDR corresponding to the order quantity for the revisiting demand can also be expressed as follows:

$$y_t{}^{\mathrm{LDR}}(\tilde{\boldsymbol{\rho}}^{t-1}) = y_t^0 + \sum_{j\in\mathcal{M}_t}\sum_{i\in\mathcal{N}_j} y_t^{i,j}\tilde{\rho}_i^j \tag{3.7}$$

That is, the order decision is based on the observed information available at the beginning of the period t.

Remark 1. *The inventory levels u_{t+1} and v_{t+1} also take an affine structure with respect to $\tilde{\boldsymbol{\rho}}^t$ as follows:*

$$u_{t+1}(\tilde{\boldsymbol{\rho}}^t) = u_{t+1}^0 + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u_{t+1}^{i,j}\tilde{\rho}_i^j \tag{3.8}$$

$$v_{t+1}(\tilde{\boldsymbol{\rho}}^t) = v_{t+1}^0 + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} v_{t+1}^{i,j}\tilde{\rho}_i^j \tag{3.9}$$

It is easy to show that inventory levels in (3.8) and (3.9) also feature the affine function of the uncertainty factors $\tilde{\boldsymbol{\rho}}^t$. This function can be derived with the closed-form expression of the balance equations as follows:

$$
\begin{aligned}
u_{t+1}(\tilde{\boldsymbol{\rho}}^t) &= u_1 + \sum_{k=1}^{t} x_k(\tilde{\boldsymbol{\rho}}^{k-1}) - \sum_{k=1}^{t} d_k \\
v_{t+1}(\tilde{\boldsymbol{\rho}}^t) &= v_1 + \sum_{k=1}^{t} x_k(\tilde{\boldsymbol{\rho}}^{k-1}) - \sum_{k=1}^{t} \tilde{\xi}_k(\tilde{\boldsymbol{\rho}}^k)
\end{aligned}
$$

As a result, the two types of decision variables related to the inventory level also take the non-anticipative property.

Upper bound of the expected positive parts

We assumed that the inventory manager decides on the order quantity based on stochastic demand in the absence of full information. As shown in the stochastic optimization model (3.4), the objective function includes the purchasing, inventory holding, and backlog costs over the entire planning horizon. For the purchasing cost, the expected value can be obtained as follows:

$$
\begin{aligned}
&\mathbb{E}\left(c_t(x_t(\tilde{\boldsymbol{\rho}}^{t-1}) + y_t(\tilde{\boldsymbol{\rho}}^{t-1}))\right) \\
&= \mathbb{E}\left(c_t(x_t^0 + \sum_{j \in \mathcal{M}_t} \sum_{i \in \mathcal{N}_j} x_t^{i,j} \tilde{\rho}_i^j + y_t^0 + \sum_{j \in \mathcal{M}_t} \sum_{i \in \mathcal{N}_j} y_t^{i,j} \tilde{\rho}_i^j)\right) \\
&= c_t(x_t^0 + y_t^0) + c_t \mathbb{E}\left(\sum_{j \in \mathcal{M}_t} \sum_{i \in \mathcal{N}_j} (x_t^{i,j} + y_t^{i,j}) \tilde{\rho}_i^j\right) \\
&= c_t(x_t^0 + y_t^0) + c_t \sum_{j \in \mathcal{M}_t} \sum_{i \in \mathcal{N}_j} (x_t^{i,j} + y_t^{i,j}) \mu_i^j \\
&\text{where } \mu_i^j = \mathbb{E}[\tilde{\rho}_i^j]
\end{aligned}
$$

In the case of the inventory holding cost, we approximated the upper bound of the expectation of the positive parts by adapting the work of Chen and Sim [26], who derived the distributionally robust bound based on the following theorem:

Theorem 1. (Chen and Sim, 2009 ([26])) *If uncertainty factors are zero-mean random variables with the positive definite covariance matrix under the support set* $\mathbf{W}$ *which is second-order conic representable, the upper bound of* $\mathbb{E}((y_0+\mathbf{y}'\tilde{\mathbf{z}})^+)$*, which is represented by* $\pi(y_0, \mathbf{y})$*, can be obtained through the optimization problem as follows:*

$$
\begin{aligned}
\pi(y_0, \mathbf{y}) = \min \quad & r_1 + r_2 + r_3 + r_4 + r_5 \\
\text{s.t.} \quad & y_{10} + \max_{\tilde{\mathbf{z}} \in \mathbf{W}} \tilde{\mathbf{z}}' \mathbf{y_1} \leq r_1 \\
& r_1 \geq 0 \\
& \max_{\tilde{\mathbf{z}} \in \mathbf{W}} \tilde{\mathbf{z}}'(-\mathbf{y_2}) \leq r_2 \\
& y_{20} \leq r_2 \\
& \frac{1}{2} y_{30} + \frac{1}{2} \left| y_{30}, \textstyle\sum^{1/2} \mathbf{y_3} \right|_2 \leq r_3 \\
& \inf_{\mu > 0} \frac{\mu}{e} exp \left(\frac{y_{40}}{\mu} + \frac{|\mathbf{u}|_2^2}{2\mu^2} \right) \leq r_4 \\
& u_j \geq p_j y_{4j} && j \in \{ j : p_j < \infty \} \\
& y_{4j} \leq 0 && j \in \{ j : p_j = \infty \} \qquad (3.10) \\
& u_j \geq -q_j y_{4j} && j \in \{ j : q_j < \infty \} \\
& y_{50} + \inf_{\mu > 0} \frac{\mu}{e} exp \left(-\frac{y_{40}}{\mu} + \frac{|\mathbf{v}|_2^2}{2\mu^2} \right) \leq r_5 \\
& v_j \geq q_j y_{5j} && j \in \{ j : q_j < \infty \} \\
& y_{5j} \leq 0 && j \in \{ j : q_j = \infty \} \\
& v_j \geq -p_j y_{5j} && j \in \{ j : p_j < \infty \} \\
& y_{10} + y_{20} + y_{30} + y_{40} + y_{50} = y_0 \\
& \mathbf{y_1} + \mathbf{y_2} + \mathbf{y_3} + \mathbf{y_4} + \mathbf{y_5} = \mathbf{y} \\
& r_i, y_{i0} \in \mathcal{R}, \quad \mathbf{y}_i \in \mathcal{R}^N, \quad i = 1, \ldots, 5 \\
& \mathbf{u}, \mathbf{v} \in \mathcal{R}^N
\end{aligned}
$$

The most distinctive difference between the model in this research and that of Chen and Sim [26] is the structure of the uncertainty set. In their work, each predefined uncertainty factor $\tilde{z}$ belongs to $\mathbf{W}$ which can be correlated but unconstrained over the period. In this study, the sum of the uncertainty factors in a particular interval is constrained. Also, uncertainty factors are not zero-mean random variables. The optimization problem (3.10) was derived based on zero-mean random variables and support set $\mathbf{W}$. However, we derived the upper bound of the expected positive parts based on non zero-mean random variables with support set $\boldsymbol{\Xi}$.

Remark 2. *The reasonable upper bound can be obtained without considering the information of the directional deviations which are related* r_4 *and* r_5 ([88]). *That is, the upper bound can be achieved even if* p_j *and* q_j *are set to* ∞.

In this study, we derived the three upper bounds of the expected positive parts related to excess inventories as follows:

$$
\begin{aligned}
&\mathbb{E}\left((u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}} \sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i)^+ \right) \\
&\leq \left(u^0_{t+1} + \max_{\tilde{\boldsymbol{\rho}}\in\boldsymbol{\Xi}} \sum_{j\in\mathcal{M}_{t+1}} \sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i \right)^+ \\
&= \pi^1(u^0_{t+1}, \mathbf{u}_{t+1})
\end{aligned}
$$

The second upper bound can be derived by using the equality $a^+ = a + (-a)^+$. Recall that the supports of the uncertainty factors are defined in $[0, 1]$. Accordingly, the value of the expectation is not canceled by zero mean as shown in previous relevant research ([26, 88]).

$$\begin{aligned}
&\mathbb{E}\left((u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i)^+\right)\\
&= \mathbb{E}\left(u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right) + \mathbb{E}\left((-u^0_{2,t+1} - \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i)^+\right)\\
&= u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} + \mathbb{E}\left((-u^0_{t+1} - \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i)^+\right)\\
&\leq u^0_{t+1} + \textstyle\sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} + \mathbb{E}\left((-u^0_{t+1} + \max\limits_{\tilde{\boldsymbol{\rho}}\in\boldsymbol{\Xi}} \textstyle\sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i)^+\right)\\
&= u^0_{t+1} + \textstyle\sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} + (-u^0_{t+1} + \max\limits_{\tilde{\boldsymbol{\rho}}\in\boldsymbol{\Xi}} \textstyle\sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i)^+\\
&= \pi^2(u^0_{t+1}, \mathbf{u}_{t+1})
\end{aligned}$$

The third upper bound can be derived by using the equality $a^+ = (a + |a|)/2$. As with the second upper bound, the value of the expectation is not canceled by zero mean, as shown in the following:

$$
\begin{aligned}
&\mathbb{E}\left((u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i)^+\right) \\
&= \frac{1}{2}\mathbb{E}\left(u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right) + \frac{1}{2}\mathbb{E}\left|u^0_{t+1} + \textstyle\sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right| \\
&\le \frac{1}{2}u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} + \frac{1}{2}\sqrt{\mathbb{E}[(u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i)^2]} \\
&= \frac{1}{2}u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} \\
&+ \frac{1}{2}\sqrt{(u^0_{t+1})^2 + 2u^0_{t+1}\sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} + \mathbb{E}\left((\sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i)^2\right)} \\
&= \pi^3(u^0_{t+1}, \mathbf{u}_{t+1})
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathbb{E}\left((\sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i)^2\right) \\
&= \sum_{j,k\in\mathcal{M}_{t+1}}\sum_{i,l\in\mathcal{N}_j}\left((u^{i,j}_{t+1})^2(\sigma_{\tilde{\rho}^j_i\tilde{\rho}^j_i} + (\mu^j_i)^2) + 2u^{i,j}_{t+1}u^{l,k}_{t+1}(\sigma_{\tilde{\rho}^j_i\tilde{\rho}^k_l} + \mu^j_i\mu^k_l) + (u^{l,k}_{t+1})^2(\sigma_{\tilde{\rho}^k_l\tilde{\rho}^k_l} + (\mu^k_l)^2)\right)
\end{aligned}
$$

and σ indicates the covariance of the uncertainty factors.

By minimizing the three bounds, $\pi^1(u^0_{t+1}, \mathbf{u}_{t+1})$, $\pi^2(u^0_{t+1}, \mathbf{u}_{t+1})$, and $\pi^3(u^0_{t+1}, \mathbf{u}_{t+1})$, in the following optimization problem (3.11), the tightest upper bound $\pi(u^0_{t+1}, \mathbf{u}_{t+1})$

can be obtained.

$$
\begin{aligned}
\pi(u^0_{t+1}, \mathbf{u}_{t+1}) \triangleq \min & \sum_{i=1}^{3} \pi^i(u^0_{i,t+1}, \mathbf{u}_{i,t+1}) \\
\text{s.t.} & \sum_{i=1}^{3} u^0_{i,t+1} = u^0_{t+1} \\
& \sum_{i=1}^{3} \mathbf{u}_{i,t+1} = \mathbf{u}_{t+1}
\end{aligned}
\tag{3.11}
$$

To retain tractability in solving the optimization problem (3.11), Assumption A is required. Otherwise, both the problem (3.11) and the robust counterpart become intractable.

Assumption A. *Uncertainty factors, $\tilde{\boldsymbol{\rho}}$ representing the uncertain revisiting rates, are the random variables whose distributions are not known but moment-based information, including, mean, covariance, and support, is assumed to be estimated. Each uncertainty factor $\tilde{\boldsymbol{\rho}}$ is distributed in the particular intervals in each period, but constrained over the period. Accordingly, uncertainty factor $\tilde{\boldsymbol{\rho}}$ lies in a support set $\boldsymbol{\Xi}$, which is a polyhedron set as shown in* (3.2).

If only the support set is considered without information of mean and covariance of uncertainty factor, the problem reduces to the robust optimization problem ([3]). By adopting the work of Chen and Sim [26], the optimization problem (3.11) for every t-th period ($t \in \mathfrak{T}^-$) can be expressed as the epigraph form as follows:

$$
\begin{aligned}
\pi(u^0_{t+1}, \mathbf{u}_{t+1}) = \min \quad & r_{1,t+1} + r_{2,t+1} + r_{3,t+1} \\
\text{s.t.} \quad & u^0_{1,t+1} + \max_{\tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi}} \tilde{\boldsymbol{\rho}}' \mathbf{u}_{1,t+1} \leq r_{1,t+1} \\
& r_{1,t+1} \geq 0 \\
& \sum_{j \in \mathcal{M}_{t+1}} \sum_{i \in \mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} + \max_{\tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi}} \tilde{\boldsymbol{\rho}}'(-\mathbf{u}_{2,t+1}) \leq r_{2,t+1} \\
& u^0_{2,t+1} + \sum_{j \in \mathcal{M}_{t+1}} \sum_{i \in \mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} \leq r_{2,t+1} \\
& \tfrac{1}{2} u^0_{3,t+1} + \tfrac{1}{2} \left| u^0_{3,t+1}, \textstyle\sum^{1/2} \mathbf{u}_{3,t+1} \right|_2 \leq r_{3,t+1} \\
& u^0_{1,t+1} + u^0_{2,t+1} + u^0_{3,t+1} = u^0_{t+1} \\
& \mathbf{u}_{1,t+1} + \mathbf{u}_{2,t+1} + \mathbf{u}_{3,t+1} = \mathbf{u}_{t+1} \\
& r_{i,t+1}, u^0_{i,t+1} \in \mathcal{R}, \mathbf{u}_{i,t+1} \in \mathcal{R}^{T \times T} \quad i = 1, 2, \text{and } 3
\end{aligned}
\tag{3.12}
$$

According to See and Sim [88], $\pi(\cdot, \cdot)$ in the optimization problem (3.10) is not exactly second-order cone representable because of the infimum term $\left(\inf_{\mu>0} \frac{\mu}{e} exp(\cdot)\right)$. However, the infimum term becomes redundant in this model because we assume p_j and q_j as ∞. If the constraints associated with r_1 and r_2, which still contain the uncertainty factors, are well defined as a robust counterpart, the remaining terms associated with the upper bound are all second-order cones. By replacing the $\max(\cdot)$ term with the dual linear program, we can derive the robust counterpart. Consider $\max\limits_{\tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi}} \tilde{\boldsymbol{\rho}}' \mathbf{u}_{1,t+1}$ in the first constraint. As shown in (3.2), uncertainty factors $\tilde{\boldsymbol{\rho}}^t$ in this model feature the polyhedron structure. For every period, $t \in \mathfrak{T}^-$, we have the following inner optimization problem:

$$\begin{aligned}
\max \quad & \sum_{j \in \mathcal{M}_{t+1}} \sum_{i \in \mathcal{N}_j} u_{t+1}^{i,j} \tilde{\rho}_i^j \\
\text{s.t.} \quad & \textstyle\sum_{j=i}^{\min(i+\tau-1,t)} \tilde{\rho}_i^j \leq 1 & i \in \{1, \cdots, t\}; \qquad (3.13)\\
& 0 \leq \tilde{\rho}_i^j \leq 1 & i \in \mathcal{N}_j, j \in \mathcal{M}_{t+1}; \\
& \tilde{\rho}_i^j = 0 & i \in \{i \mid i + \tau \leq j\}, j \in \{\tau+1, \cdots, t\};
\end{aligned}$$

By strong duality, each inner optimization problem for every period ($t \in \mathfrak{T}^-$) can be reformulated as a dual linear program as follows:

$$\begin{aligned}
\min \quad & \sum_{j \in \mathcal{M}_{t+1}} \sum_{i \in \mathcal{N}_j} \alpha_{t+1}^{i,j} + \sum_{i=1}^{t} \beta_{t+1}^{i} \\
\text{s.t.} \quad & \alpha_{t+1}^{i,j} + \beta_{t+1}^{i} \geq u_{t+1}^{i,j} & i \in \mathcal{N}_j, j \in \mathcal{M}_{t+1}; \\
& \alpha_{t+1}^{i,j} \geq 0 & i \in \mathcal{N}_j, j \in \mathcal{M}_{t+1}; \qquad (3.14)\\
& \beta_{t+1}^{i} \geq 0 & i \in \{i \mid i \leq t\}; \\
& \alpha_{t+1}^{i,j} = 0 & i \in \{i \mid i + \tau \leq j\}, j \in \{\tau+1, \ldots, t\};
\end{aligned}$$

where $\alpha_t^{i,j}$ and β_t^i are the dual variables of each constraint in (3.13), respectively.

By replacing the $\max(\cdot)$ term with the dual linear program presented in (3.14), the robust counterpart can be achieved. With the same manner, $\max(\cdot)$ term in the constraint related to r_2 is also reformulated to the robust counterpart. By solving the robust counterpart of the optimization problem (3.12), the tighter upper bound can be achieved than that of each bound.

$$\mathbb{E}\left((u_{t+1}^0 + \sum_{j \in \mathcal{M}_{t+1}} \sum_{i \in \mathcal{N}_j} u_{t+1}^{i,j} \tilde{\rho}_i^j)^+ \right) \le \pi(u_{t+1}^0, \mathbf{u}_{t+1}) \le \min_{i=1,2,3} \pi^i(u_{t+1}^0, \mathbf{u}_{t+1})$$

The upper bound of the expected costs related to backlogged inventories can be derived similarly as follows:

$$\mathbb{E}\left((u_{t+1}^0 + \sum_{j \in \mathcal{M}_{t+1}} \sum_{i \in \mathcal{N}_j} u_{t+1}^{i,j} \tilde{\rho}_i^j)^- \right) \le \pi(-u_{t+1}^0, -\mathbf{u}_{t+1}) \le \min_{i=1,2,3} \pi^i(-u_{t+1}^0, -\mathbf{u}_{t+1})$$

Robust counterpart of the IMMOR

Based on the LDR, the robust counterpart of the IMMOR (RIMMOR) can be formulated as follows:

$$
\begin{aligned}
\min \quad & \sum_{t\in\mathfrak{T}^-} [c_t(x_t^0+y_t^0) + c_t(\sum_{j\in\mathcal{M}_t}\sum_{i\in\mathcal{N}_j}(x_t^{i,j}+y_t^{i,j})\mu_i^j) + h_t\pi(u_{t+1}^0, \mathbf{u}_{t+1}^{i,j}) \\
& + h_t\pi(v_{t+1}^0, \mathbf{v}_{t+1}^{i,j}) + b_t\pi(-u_{t+1}^0, -\mathbf{u}_{t+1}^{i,j}) + p_t\pi(-v_{t+1}^0, -\mathbf{v}_{t+1}^{i,j})] \\
\text{s.t.} \quad & u_{t+1}^0 = u_t^0 + x_t^0 - d_t \qquad t\in\mathfrak{T}^-; \\
& u_{t+1}^{i,j} = u_t^{i,j} + x_t^{i,j} \qquad t\in\mathfrak{T}^-, i\in\mathcal{N}_j, j\in\mathcal{M}_t; \\
& v_{t+1}^0 = v_t^0 + y_t^0 \qquad t\in\mathfrak{T}^-; \\
& u_{t+1}^{i,j} = \begin{cases} -d_i, & t\in\mathfrak{T}^-, j=t, j-\tau+1\le i\le j; \\ v_t^{i,j}+y_t^{i,j} & t\in\mathfrak{T}^-, i\in\mathcal{N}_j, j\in\mathcal{M}_t; \end{cases} \\
& x_t^0 + \mathbf{x_t}'\tilde{\boldsymbol{\rho}} + y_t^0 + \mathbf{y_t}'\tilde{\boldsymbol{\rho}} \le K_t \qquad t\in\mathfrak{T}^-, \tilde{\boldsymbol{\rho}}\in\boldsymbol{\Xi}; \\
& x_t^0 + \mathbf{x_t}'\tilde{\boldsymbol{\rho}} \ge 0 \qquad t\in\mathfrak{T}^-, \tilde{\boldsymbol{\rho}}\in\boldsymbol{\Xi}; \\
& y_t^0 + \mathbf{y_t}'\tilde{\boldsymbol{\rho}} \ge 0 \qquad t\in\mathfrak{T}^-, \tilde{\boldsymbol{\rho}}\in\boldsymbol{\Xi};
\end{aligned}
\tag{3.15}
$$

Remark 3. *RIMMOR does not need non-anticipative constraints such as,* $u_{t+1}^{i,j}=0$, $v_{t+1}^{i,j}=0$ $(t\in\mathfrak{T}^-,\ i\in\mathcal{N}_j,\ j\in\mathcal{M}_{t+1})$ *or* $x_t^{i,j}=0$, $y_t^{i,j}=0$ $(t\in\mathfrak{T}^-, i\in\mathcal{N}_j, j\in\mathcal{M}_t)$. *Equations* (3.6) – (3.9) *already incorporate the non-anticipative property by summing up the decision variables until the available uncertainty factors in each period* t.

For the constraint representing the capacity of the order quantity, the uncertainty factors also remain. Therefore, we reformulated the constraint in each period

$(t \in \mathfrak{T}^-)$ as the robust counterpart in the same manner from (3.13) to (3.14):

$$
\begin{aligned}
& x_t^0 + \mathbf{x_t}'\tilde{\boldsymbol{\rho}} + y_t^0 + \mathbf{y_t}'\tilde{\boldsymbol{\rho}} \le K_t, \qquad \tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi} \\
& \iff x_t^0 + y_t^0 + \max_{\tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi}} \left(\sum_{j \in \mathcal{M}_t} \sum_{i \in \mathcal{N}_j} (x_t^{i,j} + y_t^{i,j}) \tilde{\rho}_i^j \right) \le K_t \\
& \iff \begin{cases}
x_t^0 + y_t^0 + \sum_{j \in \mathcal{M}_t} \sum_{i \in \mathcal{N}_j} \theta_t^{i,j} + \sum_{i=1}^{t-1} \delta_t^i \le K_t & \\
\theta_t^{i,j} + \delta_t^i \ge x_t^{i,j} + y_t^{i,j} & i \in \mathcal{N}_j, j \in \mathcal{M}_t; \\
\theta_t^{i,j} \ge 0 & i \in \mathcal{N}_j, j \in \mathcal{M}_t; \\
\delta_t^i \ge 0 & i \in \{i \mid i \le t-1\}; \\
\theta_t^{i,j} = 0 & i \in \{i \mid i + \tau \le j\}, j \in \{\tau+1, \ldots, t-1\};
\end{cases}
\end{aligned}
$$

where $\theta_t^{i,j}$ and δ_t^i are the dual variables.

In the cases of constraints related to non-negative conditions of decision variables, the constraints of all periods $(t \in \mathfrak{T}^-)$ can also be reformulated by defining the inner optimization problems as follows:

$$
\begin{cases}
x_t^0 + \mathbf{x_t}'\tilde{\boldsymbol{\rho}} \ge 0, & \tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi} \\
y_t^0 + \mathbf{y_t}'\tilde{\boldsymbol{\rho}} \ge 0, & \tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi}
\end{cases}
\iff
\begin{cases}
x_t^0 - \max_{\tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi}} \sum_{j \in \mathcal{M}_t} \sum_{i \in \mathcal{N}_j} x_t^{i,j} \tilde{\rho}_i^j \ge 0 \\
y_t^0 - \max_{\tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi}} \sum_{j \in \mathcal{M}_t} \sum_{i \in \mathcal{N}_j} y_t^{i,j} \tilde{\rho}_i^j \ge 0
\end{cases}
$$

By developing the dual linear program and substituting it for the inner optimization problem, the robust counterpart can be derived. We omit the expression of the robust counterpart which has the same process from (3.13) to (3.14). As a result, the deterministic second-order cone program, which is solvable with the interior-point method algorithm within the polynomial time, was derived from the multistage stochastic optimization model ([2]). We provide a small-size numerical example in Appendix B.1 to make it easier for readers to understand.

Relation to the restricted linear decision rule

Recall that Figure 3.2 represents the coefficient matrix of the revisiting rate. When the duration of the entire period T is relatively larger than the expiry date τ, most of the coefficients are zero. Accordingly, the inventory balance equation associated with revisiting rates whose values are zero does not have an effect. In other words, the parts where the values of $\tilde{\boldsymbol{\rho}}^t$ are zero do not directly affect the inventory level, leaving only the balance equation between the relevant decision variables. By forcing the decision variables of these parts to zero, the solution space could be reduced, which helps the commercial optimization solver to find a solution efficiently. In this manner, Ang et al. [3] proposed a *restricted linear decision rule* (RLDR).

Proposition 5. *In this model, the objective value obtained by the RLDR provides the same objective value as the LDR, which was known to provide an inferior solution from the robust counterpart model in Ang et al.* [3].

We will support Proposition 5 through an example. Consider the problem with planning horizon $t \in \{1, \ldots, 8\}$ and an expiry date τ as 3. Consequently, some revisiting rates, such as $(\tilde{\rho}_1^4, \tilde{\rho}_1^5, \tilde{\rho}_2^5, \cdots)$, become zero. For simplicity, consider only the balance equation for $\tilde{\rho}_1^4$ at $t = 6$. According to the LDR, the robust counterpart includes the equation $v_6^{1,4} = v_5^{1,4} + y_5^{1,4}$ in the balance equation $v_6^0 + \sum_{j=1}^{5} \sum_{i:i \leq j} v_6^{i,j} \tilde{\rho}_i^j = v_5^0 + \sum_{j=1}^{4} \sum_{i:i \leq j} v_5^{i,j} \tilde{\rho}_i^j + y_5^0 + \sum_{j=1}^{4} \sum_{i:i \leq j} y_5^{i,j} \tilde{\rho}_i^j - d_3 \tilde{\rho}_3^5 - d_4 \tilde{\rho}_4^5 - d_5 \tilde{\rho}_5^5$. As presented in the balance equation, $v_6^{1,4} = v_5^{1,4} + y_5^{1,4}$ does not affect the inventory level. Restricting the relevant decision variables $y_5^{1,4}$ to zero can allow the problem to be solved efficiently while retaining the objective value.

3.4 Computational experiments

In this section, we describe the results of three types of computational experiments. The experiments were conducted to answer the following research questions:

(i) Does RIMMOR, which constrains the sum of the uncertainty factors over the period to less than or equal to 1, retain tractability until a modest data size, as the model of See and Sim [88] does?

(ii) How much robustness does RIMMOR guarantee when random demand is realized compared to the deterministic model, which estimates the uncertainty factors?

(iii) Depending on the propensity of the customer, what tendency does the inventory policy of RIMMOR show?

(iv) What tendency does total cost show when the expiry date varies?

Research questions (i), (ii), and (iii) and (iv), are answered by Experiments (1) – (3), respectively. Results of Experiments (1) – (3) and analyses are described in Section 4.1 – 4.3. All computational experiments were conducted by FICO XPRESS-IVE version 7.2 with an Intel® Core$^{\mathrm{TM}}$ i5-7400 CPU @ 3.0 GHz.

RIMMOR needs the mean and covariance of the uncertainty factors. Most of the previous studies that considered a factor-based demand model assumed the uncertainty factors as zero-mean random variables and unconstrained. Accordingly, the mean and covariance could be easily derived. In the RIMMOR, however, each uncertainty factor has support between 0 and 1, and the sum of uncertainty factors within a certain interval is less than or equal to 1. This makes deriving an accurate

Table 3.2: Results of Experiment 1 which was conducted to verify the tractability of the RIMMOR

	Data (total planning horizon_expiry date) when order capacity is 350							
	20_5	20_10	25_5	25_10	25_15	30_10	35_5	35_8
Objective value	34119.7	39939.3	31678.9	39706.8	45298.7	47253.3	54248.6	62904.0
Computation time (s)	22.9	59.6	71.4	277.5	499.7	866.5	592.4	1434.5

mean and covariance difficult. Therefore, we estimated the mean and covariance through data sampling with 10,000 iterations. Pseudocode for the generation of the uncertainty factors is described in Appendix B.2. We assume that all customers revisit the store because we want to observe protection against the worst case. Thus, we made the sum to be 1 by forcing the last iteration of Algorithm 1.

3.4.1 Experiment 1: tractability of the RIMMOR

We conducted Experiment 1 to investigate the tractability of the RIMMOR. Experiment 1 was conducted by varying the planning horizon and expiry date. The sample mean and covariance were estimated from data generated through Algorithm 1. The results of Experiment 1 are presented in Table 3.2. As can be seen from Table 3.2, when the planning horizon increased, the computation time also increased. Furthermore, as the expiry date increased, the computation time increased. Nevertheless, the RIMMOR was tractable until a modest data size. In practice, a retailer who runs the MOR application in a convenience store has one order cycle per day. With the RIMMOR, the retailer can establish a one-month plan for the BOGO promotion.

3.4.2 Experiment 2: robustness of the RIMMOR

The solution obtained through the RIMMOR is a decision rule for an order quantity. Thus, solving the optimization problem does not provide the order quantity for each period but establishes a policy. To figure out how the decision rule guarantees the protection of the realized uncertain data, we conducted comparative experiments. For the comparison group, simulation, we assume that the inventory manager regards the uncertainty factors as deterministic values by estimating the mean based on the data from Algorithm 1. Accordingly, uncertain demands were set as deterministic values for the entire period. The order quantities were obtained by solving the deterministic model (2). In this manner, simulation results and policies from the LDR were compared through 10,000 iterations of the experiment. A summary of the results is illustrated in Figure 3.6.

As we can see from Figure 3.6, the robustness of RIMMOR was guaranteed compared to the simulation experiments. Although the objective values of the LDR were worse than the best case of the deterministic model, the RIMMOR showed overwhelmingly better results for the worst case. The most important thing to recognize is that the RIMMOR provided stable solutions in terms of the fluctuation. Even though uncertainty factors can be realized with any value, the difference between the minimum and maximum objective values was not significant.

3.4.3 Experiment 3: effect of duration of the expiry date under the different customers' revisiting propensities

We conducted Experiment 3 to explore the effects of the duration of the expiry date and customers' revisiting propensities on the total cost. One of the research questions

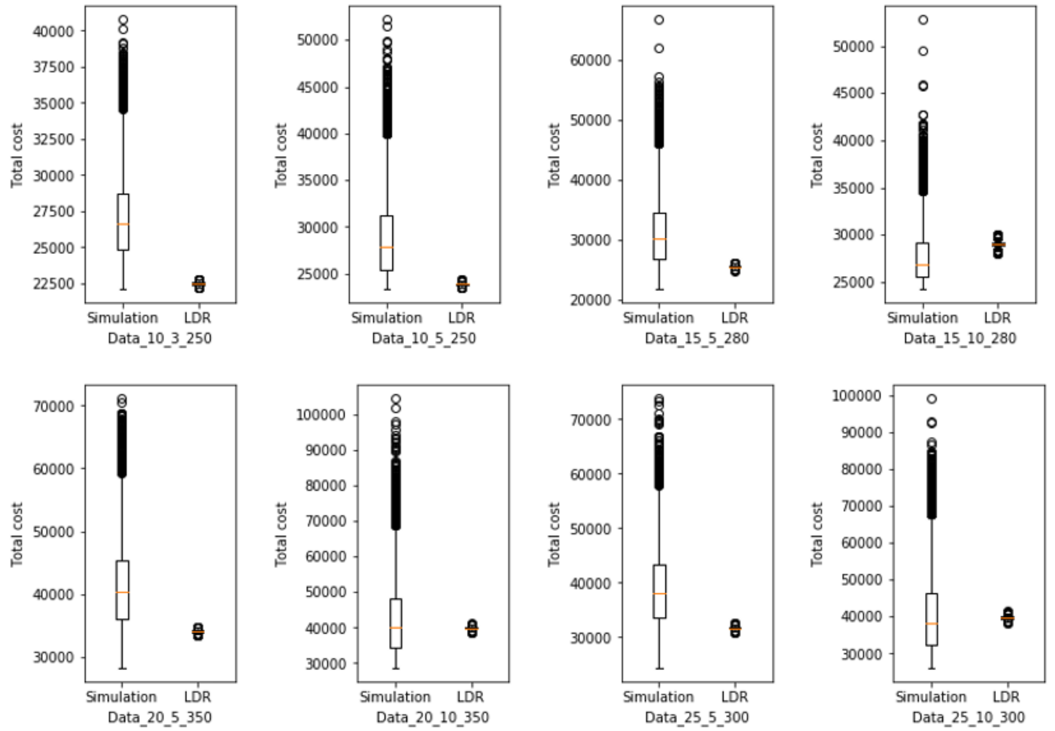


Figure 3.6: Results of Experiment 2, protection against realized uncertainty factors

was how the objective value changes by varying the expiry date. We could make two conflicting inferences at the same time. First, we expected that the total expectation cost would be lowered by the smoothing effect when the expiry date becomes longer. Second, we thought that a larger order quantity should be replenished to cope with the worst-case scenario, which would incur a higher cost. We also thought that customers' revisiting tendencies might affect the total cost. Therefore, Experiment 3 was conducted by varying the expiry date τ according to three types of customers: (i) a *general customer* (GC) who was already considered in the previous subsection; (ii) an *impetuous customer* (IC), who has a high revisiting rate near the purchasing date; and (iii) a *procrastinating customer* (PC), who has a high revisiting rate

Table 3.3: Results of Experiment 3 which was conducted by varying the duration of the expiry date under the different customers' revisiting propensities

		Data (total planning horizon_expiry date_capacity)					
Customer type	Policy	10_2_280	10_3_280	10_4_280	10_5_280	10_6_280	10_7_280
IC	LDR	14826.8	15211.0	15515.7	15452.3	15473.0	15480.8
	EV|PI	14256.0	14410.4	13507.6	14535.7	14550.8	14558.8
GC	LDR	14827.6	15588.6	16254.3	16810.8	17254.4	17576.3
	EV|PI	14066.3	14431.9	15050.7	15595.7	16032.2	16348.7
PC	LDR	14827.3	15974.8	17133.6	18176.5	18969.5	19417.7
	EV|PI	14256.3	14836.5	15904.1	16931.5	17776.5	18395.5

when the expiry date approaches. For IC, we assumed that the two products are most likely to be taken on the purchasing date and the revisiting rate decreases as the expiry date approaches. In the case of PC, it is assumed that the revisiting rate increases the further the expiry date from the purchasing date. Data generations for IC and PC are described in Algorithms 2 and 3 in Appendix B.3. Using the generated data, we conducted Experiment 3 to explore how the objective value varies according to the duration of the expiry date and customer type. To demonstrate the validity of the objective value, $expected\ value\ given\ perfect\ information$ (EV|PI) was introduced to substitute the multistage stochastic optimization model (3.4). Since the model (3.4) is difficult to solve directly, we solved each of the 10,000 sets of randomly generated uncertainty factors through the deterministic model (3.3) and considered the average value as EV|PI. EV|PI was assumed that all information about the distribution of uncertainty factors is known to the inventory manager. Accordingly, EV|PI was used to validate the objective value from the RIMMOR. The results of Experiment 3 are represented in Table 3.3 and illustrated in Figure 3.7. From Table 3.3 and Figure 3.7, we derived the following observations.

Observation 5. *In the case of IC, the objective value of the LDR did not show*

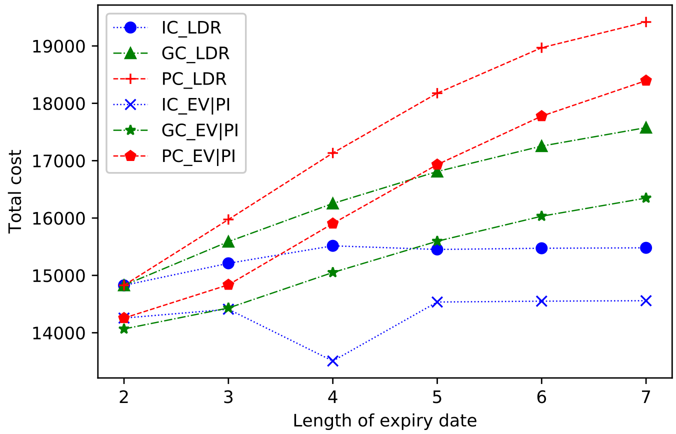


Figure 3.7: Comparing between IC, GC, and PC by varying the expiry date τ in Experiment 3

much variability even when τ increased. In contrast, the objective values of GC and PC show an increasing tendency when τ increases.

In the case of IC, sampling data showed that the sum of uncertainty factors converged to 1 near the purchasing date. Although τ increases, most of the revisiting demands near the expiry date were 0. Accordingly, extending the expiry date τ did not have a significant impact and a noticeable difference was not shown in the objective values. For GC, because the revisiting rate was distributed evenly during the period, customers who have the potential to revisit until the expiry date were considered. Accordingly, more conservative solutions were obtained. In the case of PC, the greater the likelihood that the customer revisits in the distant future, the higher the revisiting demand at the end of the planning horizon. Consequently, Experiment 3 showed the results stated in Observation 1.

Observation 6. *Results of Experiment 3 show that PC $\geq$ GC $\geq$ IC for the LDR.*

In the case of PC, when τ increases, the revisiting demand accumulates at the end of the period. Consequently, more inventory is accumulated in advance to cope with cumulative revisiting demands, and a substantial penalty occurs due to the unacceptable quantity of the order. For GC, revisiting demands are spread evenly. As the expiry date τ is extended further from the purchase date, revisiting demand also accumulates at the end of the period. Accordingly, we observed that the objective value increased as τ was extended. Notable results were observed for IC. Generated data showed that the majority took the second item at the purchase date and gradually decreased from the next period. Thus, the backorder cost occurred by the order capacity was less than that of the other two cases and large variability in the objective value was not observed.

Also, as can be seen from Figure 3.7, the LDR provided a reasonable upper bound when EV|PI was regarded as a benchmark. Although IC exhibits the tendency related to the gap between LDR and EV|PI when the expiry date τ is 4, this can be interpreted as a smoothing effect of the revisiting demand in EV|PI. To sum up, the results of Experiments 2 and 3 demonstrate that RIMMOR establishes a robust and stable plan against uncertainty factors while providing a reasonable upper bound in the multistage stochastic optimization model.

3.5 Summary

MOR is an innovative application that has been downloaded by more than ten million customers. Customers using MOR can revisit the store at a later date to take the second product that they have earned through a BOGO promotion. For retailers, however, it is difficult to respond efficiently with the existing inventory model due to the high level of uncertainty with regard to the revisiting date. Accordingly, we developed the RIMMOR and demonstrated through computational experiments that it could provide a reasonable inventory policy.

The distributionally robust optimization approach presented in this study has a distinctive feature that differentiates it from previous studies. The sum of the uncertainty factors in a particular interval is constrained to less than or equal to 1. Constrained uncertainty factors from an inner optimization problem were reformulated into a dual linear program to retain robustness and tractability. The robust counterpart was developed as a second-order cone program, which was tractable until a modest data size. Moreover, the robust counterpart provided a stable solution for the worst case without full information about the distribution. Compared with the EV|PI, which is the substitute for a multistage stochastic optimization model, the robust counterpart provided the reasonable bound derived from only mean, support, and covariance of uncertainty factors.

3.5.1 Managerial insights

To provide managerial insights, we conducted computational experiments with the three types of customer tendencies. We generated three types of data to incorporate the property of GC, PC, and IC. From the results, we derived the following managerial insights for the retailer:

(i) For certain products, customers are less likely to take both items on the purchasing date and thus have a higher probability of revisiting in the future. For these products, we recommend setting the expiry date not too far in the future. Perishable items or heavy products might be examples of items that are not usually taken at once.

(ii) Products that many customers want to take at the last moment before the expiry date show the greatest cost among the three comparison experiments. This can be the case for a product that is used for a relatively long time. For the retailer, it is necessary to hold a large amount of inventory until the end of the period, but sometimes this is difficult due to the capacity of the order quantity. Therefore, we suggest developing a way to induce customers to make a reservation on the purchase date to pick up the second product on a certain date. Then, customers will not face stockouts and retailers can reduce uncertainty.

(iii) We recommend that BOGO promotions offered through MOR be conducted for daily necessities. In the case of daily necessities, there is a high possibility that customers will return soon for the second product. Even if the expiry date is set far into the future, the cost during the entire period does not change significantly.

Chapter 4

Robust Multiperiod Inventory Model Considering Refurbishment Service and Trade-in Program

4.1 Introduction

Since the introduction of *sustainable development* in 1987 and the announcement of the 1992 *Rio Declaration* on environmental protection, various efforts have been made throughout the world to preserve the environment ([18, 30]). One of the challenges to the manufacturing sector was the concept of $remanufacturing$. It refers to the process of collecting broken, discarded, or returned products and transforming them into "like-new" products through reconstruction, repair, cleaning, and repurposing. The performance and condition of the remanufactured products are similar to those of new products, but remanufacturing can lower the purchasing cost of raw materials and reduce adverse effects on the environment. By drawing on these strengths of remanufacturing, several companies that produce high-tech devices, such as mobile phones and personal computers, have recently launched a $trade-in$ $program$. As an effective sales promotion strategy, companies collect used products from customers and provide them with new-generation products at discount prices. For example, *Apple* launched the *iPhone Upgrade Program* in 2016, which, by collecting used iPhone 7 models from customers, allowed the company to offer iPhone

8 models to customers at discount prices. For the manufacturing industry, whose development cycle of new-generation products becomes ever shorter, the trade-in program plays a prominent role in customer retention and cutting down on incursion from competitors. It also mitigates the decline in sales of the new-generation product that results from the released old-generation products. Due to the trade-in program, companies can increase their customers' repeating purchases and alleviate the regret of customers who have already bought an old-generation product ([108, 109]). This successful strategy has caught the attention of researchers, including the area of management science, operations management, and supply chain management. Thus, various studies have been conducted to examine strategies for ongoing operations and to demonstrate the effects of the trade-in program.

Despite the successful introduction and operation of the trade-in program, however, little attention has been posed to the research analyzing the effect of the trade-in program on a *refurbishment service*. The refurbishment service is one of the warranty services that the customer submits the malfunctioning product for repair, and the retailer provides a *refurbished product* immediately instead of repairing that product. In general, a refurbished product refers to a product in a like-new condition, sold at a discount price. Many companies, under the refurbishment service, occasionally sell defective products, products on display, product demos, or returned products at a discount price. For most retailers nowadays and especially for larger retailers such as *Apple* or *Samsung*, the term "refurbished" refers to products that have truly been reconditioned to a like-new state, thereby sparing customers the inconvenience of waiting for repairs or other replacement hassles. In the case of the mobile phone, which customers use every day, it might be inconvenient when the repairing process takes too long time. Through the refurbishment service, customers can enjoy the same effect as repairing the product immediately, which leads to high customer satisfaction.

From the retailer's perspective, however, it is challenging to prepare sufficient refurbished products since the exchange is a one-to-one arrangement. Predicting the number of products customers will return, and matching that number with a sufficient supply of refurbished products, can be difficult if the refurbished product under warranty is out of stock. Furthermore, if the corresponding product is a discontinued model, it would be cumbersome and inefficient to ask the manufacturer to produce a discontinued product. By introducing the trade-in program, retailers can acquire additional products that are in better condition than returned malfunctioning products. This suggests that research into inventory management considering the trade-in program and refurbishment service simultaneously is required. Accordingly, we started the research from the following research questions:

(i) Could the introduction of a trade-in program not only play a role in a sales promotion that increases customer demand but also ensure the stable operation of a refurbishment service?

(ii) How should an inventory model or supply chain system that incorporates the trade-in program and refurbishment service be modeled?

(iii) How should demands in this closed-loop supply chain system be modeled if they are uncertain and correlated?

To examine the refurbishment service and trade-in program in the inventory management problem, we considered a supply chain system, including a manufacturer, remanufacturer, and retailer. The inclusion of a remanufacturer shifts the system from an $open - loop\ supply\ chain\ system$ to a $closed - loop\ supply\ chain\ system$ ([39]). In the closed-loop supply chain system, there exists a reverse flow of returned products from customers to the retailer or remanufacturer. According to Souza [97], the type of product returned can be classified into three categories in the closed-loop system; $consumer\ returns$, $end - of - use\ returns$, and $end - of - life\ returns$. In the case of consumer returns, the returned product is rarely used and is repurposed mostly through a refurbishment process. In the case of end-of-use returns, the product has been used sufficiently, but defects are difficult to notice, and the product can be submitted to a trade-in program. For end-of-life returns, the product has reached the end of its useful life; that is, it cannot adequately perform its function. Although it can be remanufactured, the process requires a relatively high cost and a long processing period compared to the work required by a product from an end-of-use return. Meanwhile, the terms $remanufacturing$ and $refurbishing$ often have been used in remanufacturing contexts without discrimination. Throughout this chapter, we clearly distinguish between the two terms, using remanufacturing to refer to the process of turning end-of-life products into like-new products and using refurbishing to refer to the process of turning end-of-use products into like-new products.

Before exploring the effect of the trade-in program on the refurbishment service, the company's decision level should be specified. According to Souza [99], decisions made along the lines of a closed-loop system can be divided into three levels, which are *strategic*, *tactical*, and *operational* levels. Strategic-level decisions are made for long-term planning purposes, such as network-design or contracts of the supply chain. Tactical-level decisions are made for mid-term planning purposes, such as inventory policies. Operational-level decisions are made for short-term planning purposes, such as scheduling, lot-sizing, or routing in the manufacturing or remanufacturing plant. Our focus in this study is on a company's (retailer's) acquisition of returned products and its return policy, which is at the tactical level. In other words, we focus on an inventory policy, which incorporates both refurbishment service and trade-in program at the time.

The service level of the refurbishment process can be expected to increase with the introduction of the trade-in program. However, decisions on replenishment and inventory control can be complicated, and uncertainty in the system can increase. As Tang and Li [104] pointed out, uncertainties inherent in the closed-loop system can include the proper correlation between demand and return, the quality of returned products, and the several parameters in the production planning. Since the main purpose of this study is to analyze the refurbishment service and trade-in program at the tactical level, we focus on the correlations between demand and return. We consider three types of demands, which are for new-generation products, refurbishment service, and the trade-in program. For the refurbishment service and trade-in program, the customers return the used product to the retailer. The structure of demand and return, which are correlated in this manner, implies that a new policy of inventory control is required. In this chapter, we consider the multiperiod inventory problem, which incorporates the three types of uncertain demands that are correlated. By adopting a factor-based demand model, the correlations of these uncertain demands can be characterized. Detailed explanations will be discussed in Section 4.5.1.

To sum up, we introduced the factor-based demand model in the multiperiod inventory model based on the closed-loop supply chain system. We approximated the stochastic optimization model by utilizing the distributionally robust optimization approach to tackle the intractability of the model. We also conducted various experiments to get answers from research questions (i) to (iii). Thus, the main contributions of this research can be summarized as follows:

- We developed a mathematical formulation based on the closed-loop system with the refurbishment service and trade-in program.
- We approximated the multistage stochastic optimization model to the second-order cone program that is computationally tractable.
- We found managerial insights that could be beneficial to the inventory manager of the retailer.

The remainder of this chapter is organized as follows: Previous relevant studies are investigated in Section 4.2. In Section 4.3, we describe the inventory model considering the refurbishment service and trade-in program (IMRSTIP). Section 4.4 deals with the mathematical formulation of the IMRSTIP. In Section 4.5, we present our computational experiments and analyses. In Section 4.6, we summarize the findings of this research.

4.2 Literature review

We investigated previous studies related to the trade-in program and inventory or lot-sizing problem in the closed-loop supply chain system. Especially, we described the studies considering the effect of the trade-in program or pricing strategy in the trade-in program in Section 4.2.1. In Section 4.2.2, the previous studies relevant to the operation at the tactical or operational level in the closed-loop system are described. After explaining previous relevant studies, we summed up the distinguishable features of this study in Section 4.2.3.

4.2.1 Effects of the trade-in program and strategic-level decisions for the trade-in program

Various studies have been conducted to analyze the benefits of introducing the trade-in program to the company. Rao et al. [81] demonstrated the effects of a trade-in program and claimed that introducing a trade-in program would inevitably raise the profit. Yin et al. [121] adopted the two-period dynamic game, including first and second generations, to examine the effectiveness of a trade-in program. By incorporating customers as forward-looking customers, they analyzed conditions that are beneficial to a company. They claimed that the durability of the product in a first generation, degree of the market heterogeneity, and uncertainty of the product in a second generation are the key determinants of successful trade-in program. Meanwhile, Agrawal et al. [1] studied the trade-in program that operated between an original equipment manufacturer and a third-party remanufacturer. They derived several insights by analyzing the effect of the trade-in program based on the game theory scheme. Also, their numerical analysis revealed that the trade-in program has environmental advantages. As evidenced by previous literature, the introduction of the trade-in program not only brings additional profits to a company but also has a tremendous environmental impact. Additionally, we examined existing studies on how strategic-level decisions should be made for the medium- and long-term by introducing the trade-in program.

In addition to analyzing the effects that trade-in programs have on companies, various research has analyzed how companies might operate the trade-in program successfully by establishing pricing strategies. Ray et al. [82] focused on determining an optimal price and rebate for trade-in products. The analysis was conducted by varying the price setting and customer segment for each scenario. Accordingly, the study identified the most favorable conditions of pricing strategies for each scenario. Li et al. [61] conducted research to find a more effective way to operate a trade-in program in a business-to-business context. They argued for the effectiveness of the trade-in program through customer segmentation and accurately predicting product returns in the trade-in program. The proposed method allows a company to design a segment-based trade-in policy. They claimed that predicting product returns, particularly for the initial return, plays a critical role in the successful operation of a trade-in program. Li and Xu [62] found an optimal pricing strategy from the perspective of a monopolistic manufacturer, where the trade-in program and leasing option were available. They identified the differences between a trade-in program and leasing option in terms of the return time and the profitability of the new product. Furthermore, they analyzed which policies are advantageous to a company based on the given conditions. In the meantime, Chen and Hsu [25] derived the optimal price, trade-in rebate, and the strategic choice for a company by considering the deterioration rate and the recovery cost of used goods. They addressed that the degree of the trade-in rebate increases according to the deterioration rate, and decreases in the manufacturing and remanufacturing costs. Zhu et al. [131] derived an optimal price to charge new customers and an optimal rebate to offer trade-in program customers under the duopoly situation in which one company operates the trade-in program, and another company does not operate the trade-in program. By analyzing the results from the Nash equilibrium, they identified positive impacts on the market share and the profitability of the trade-in program. Han et al. [43] investigated conditions necessary for a successful trade-in program by determining the price and production quantity. By taking into account the factors of receptivity, durability, and subsidy, they identified insights for both companies and governments to implement a trade-in program effectively.

An *omni-channel*, which integrates an online channel with the retail service, was also considered in the trade-in program context. Cao et al. [20] examined the trade-in program based on three types of distribution channels, including online, offline, and dual channels. They analyzed the condition for which channels are the best choice for retailers according to the shipping cost. Cao et al. [21] considered the trade-in program with the dual-format retailing model, including the self-run store and third-party store. Differing from the traditional trade-in program, gift cards or cash coupons were provided to customers. The introduction of a trade-in program in such a system can lower the trade-in rebate but brings significant benefits to a company. Meanwhile, Sheu and Choi [94] investigated pricing strategies under the variable of market competition. By including the concept of extended consumer responsibility in the model, they provided syncretic value-oriented prices and trade-in rebates in a trade-in program operating within a market competition setting. De Giovanni and Zaccour [29] analyzed investment decisions for quality improvement and pricing in a trade-in program. In their work, they divided returns into two categories, including the passive return and active return, and they divided the pricing strategy into two types, which remained constant or varied over time. Ma et al. [65] studied how the quality of returned products affected the trade-in program, unlike previous studies that only focused on how pricing affected the trade-in program under the remanufacturing environment. They identified the effects of these double references and derived the condition under which both the manufacturer's profits and customer surplus would benefit.

As can be seen from the literature as mentioned above, many researchers demonstrated the profitability of the trade-in program. If a long-term plan, such as a pricing strategy, is appropriately established, revenue would surely be increased. After that, planning from a mid-term perspective, such as order policy for inventory management, should be established. Unlike an existing inventory system, an inventory system that includes the trade-in program makes the supply chain a closed-loop system. Therefore, we investigated previous studies related to the inventory model based on the closed-loop supply chain system and identified the differences we found from this research.

4.2.2 Inventory or lot-sizing model in a closed-loop supply chain system

Mathematical formulations based on the closed-loop supply chain system can be categorized into two types; an *economic order quantity*, which is based on a continuous-time nonlinear program, and a mixed-integer program which is based on a discrete-time planning horizon. We mainly investigated the latter type, which is close to this study. [73] developed a mathematical formulation based on the mixed-integer program considering not only the quantities for the manufacturing and remanufacturing products but also the fixed cost for the opening decision of the plants. In the work of Li et al. [63], they introduced two types of binary variables to incorporate the fixed costs for manufacturing and remanufacturing. Gaur et al. [36] considered reconditioned products in a closed-loop system that could be involved in several stages, depending on the condition of the returned product. Mardan et al. [66] developed the multi-echelon network as the closed-loop supply chain system. To overcome the complexity, they developed the Benders decomposition algorithm and validated the performance of the algorithm.

Especially in recent years, many studies in the area of the remanufacturing have incorporated an uncertain environment. Denizel et al. [31] considered the uncertain quality of the returned products in the closed-loop supply chain. They developed a multistage stochastic optimization model by generating scenarios of possible qualities of returned products. With a focus on robust optimization, various research has been conducted by optimizing the problem against the worst-case scenario. By developing robust counterparts to existing set-ups, feasible solutions have been guaranteed under all possible scenarios. For the cases of Eslamipoor et al. [33] and Pishvaee et al. [77], they regarded the uncertain demand set as a box shape. In addition, Eslamipoor et al. [33] considered inventory-related costs as uncertainty. Meanwhile, the concept of the *budget of uncertainty* in demand, which was proposed initially by Bertsimas and Sim [14], was adopted in the closed-loop supply chain system

([54, 128, 129, 119]). In the case of Hasani et al. [44], they considered the perishable item under uncertain demand. As can be found from the above studies, the robust optimization model under the closed-loop supply chain has mainly considered the uncertain set as a box or polyhedron shape. Wei et al. [111] regarded demand as an uncertain value that belongs to the polyhedron. To make the formulation, including the uncertain parameter as a robust counterpart, they adopted the budget of uncertainty. Although the prevailing purpose of the robust optimization through the budget of uncertainty was based on a static situation, they handled the multiperiod setting by forcing the independence of parameter Γ, as Bertsimas and Thiele [15] proposed. Talaei et al. [101] focused on reducing the rate of carbon emission in the closed-loop system. They investigated the effects of uncertainties on the cost and demand rate by adopting the robust fuzzy programming approach. Jabbarzadeh et al. [51] considered the disruption risk in the closed-loop system. They developed the mathematical formulation as the stochastic robust optimization model and solved the problem through the Lagrangian relaxation. Bi-level optimization was also considered in the closed-loop supply chain network by Hassanpour et al. [45]. They considered the incentive strategy for different qualities of the returned products. The robust bi-level optimization model was developed by considering the government as a leader and supply chain designer as a follower. To solve the problem efficiently, they utilized the meta-heuristic algorithm.

As detailed in this section, plenty of studies related to the inventory or lot-sizing model under the closed-loop supply chain system were conducted. Owing to the remanufacturing process, the complexity of the model increased, and tractability issues occurred. Accordingly, many studies have been conducted to improve the solution methodology in terms of efficient computation. In addition, as Tang and Li [104] mentioned, the closed-loop supply chain can raise a variety of uncertainties. Thus, many studies considered the robust optimization approach to handle such uncertainties.

4.2.3 Distinctive features of this research

As evident from the above literature, no study has yet considered the trade-in program and refurbishment service simultaneously from a tactical level. Bian et al. [16] integrated the model based on the trade-in program with warranty service, but they focused on pricing rather than inventory control. In the case of Huang [49], the trade-in program was incorporated in the closed-loop system structure, but the refurbishment service was not considered. Meanwhile, Hong et al. [46] investigated the role of the value-added service in the closed-loop supply chain, which has a similar property to the trade-in program. They examined the effects of the value-added service on a retailer and manufacturer by analyzing the model and conducting numerical experiments. They also designed a supply chain contract based on the service cost-sharing mechanism to improve the whole system's profitability. However, this study was also oriented to make the decision based on the strategical level.

In this chapter, we focused on a multiperiod inventory model that took into account both a trade-in program and refurbishment service at the same time. To address uncertain demands in the multistage decision process, we considered an adjustable robust optimization approach. This robust optimization approach, which is based on the wait and see decision model, is more natural than the here and now decision model ([9]). In this scheme, the decision can be delayed by observing the part of uncertain factors that are realized within the course of the period. Also, no study has considered the adjustable optimization approach to the closed-loop supply chain system. To develop the closed-loop supply chain model based on the adjustable robust optimization approach, we refer to the multiperiod linear program model. We mainly referred to the models of Teunter et al. [105] and Wei et al. [111] and modified those models to fit the circumstance of this study. To emphasize the characteristics of this study from the previous studies related to the closed-loop system, we summarize the relevant literature in Table 4.1.

Table 4.1: Comparisons of this study and previous research relevant to the closed-loop system

References	Mathematical formulation	Uncertainty	Refurbishment service or Trade-in program	Solution methodology
Teunter et al. (2006) [105]	Mixed-integer program			Dynamic programming
Özceylan and Paksoy (2013) [73]	Mixed-integer program			Commercial solver
Li et al. (2014) [63]	Mixed-integer program			Meta-heuristic
Gaur et al. (2017) [36]	Mixed-integer nonlinear program			Heuristic
Huang (2018) [49]	Stackelberg game		✓	Closed-form solution
Mardan et al. (2019) [66]	Mixed-integer program			Benders decomposition
Denizel et al. (2009) [31]	Linear program	✓		Stochastic program
Wei et al. (2011) [111]	Linear program	✓		Budget of uncertainty
Pishvaee et al. (2011) [77]	Mixed-integer program	✓		Robust counterpart from box uncertainty set
Hasani et al. (2012) [44]	Mixed-integer nonlinear program	✓		Linearization
Kim et al. (2018) [54]	Mixed-integer program	✓		Budget of uncertainty
Eslamipoor et al. (2015) [33]	Mixed-integer program	✓		Robust counterpart from box uncertainty set
Talaei et al. (2016) [101]	Mixed-integer program	✓		Chance constrained fuzzy programming
Jabbarzadeh et al. (2018) [51]	Mixed-integer program	✓		Robust counterpart and Lagrangian relaxation
Zhang et al. (2019) [128]	Mixed-integer program	✓		Fuzzy theory and Robust optimization
Zhou and Sun (2019) [129]	Linear program	✓		Budget of uncertainty
Yavari and Geraeli (2019) [119]	Mixed-integer program	✓		Budget of uncertainty and Heuristic
Hassanpour et al. (2019) [45]	Bi-level mixed-integer program	✓		Robust counterpart and Meta-heuristic
This study	Linear program	✓	✓	Affinely adjustable robust optimization approach

4.3 Problem description

We consider a multiperiod inventory model based on the discrete-time planning horizon $t \in \{1, \ldots, T\}$. Let $\tilde{\zeta}_t$, $\tilde{\phi}_t$, and $\tilde{\psi}_t$ denote the uncertain demands of the new-generation product, trade-in program, and old-generation product, respectively, at period $t \in \{1, \ldots, T-1\}$. The retailer provides new-generation products to customers of the new-generation demands and trade-in program demands. Customers who purchase the new-generation product through the trade-in program return end-of-use products, which are old-generation products, to the retailer. The retailer then determines how many products to send to the remanufacturer and how many to keep in inventory. If the retailer sends the end-of-use product to the remanufacturer, the retailer receives the like-new conditioned product after a lead time L_r with paying the refurbishing cost $c_{r,t}$ per unit at period t. Otherwise, when the retailer holds the product in inventory, the inventory holding cost $h_{w,t}$ per unit occurs at period t. In the case of customers who arrive at the store to get refurbishment service, they return the end-of-life product to the retailer and receive the like-new conditioned product. The retailer also determines how many products to send for remanufacturing and how many to keep in inventory. In this case, lead time L_m takes and remanufacturing cost $c_{m,t}$ occurs per unit at period t. Otherwise, the holding cost $h_{I,t}$ occurs per unit at period t. Meanwhile, the manufacturer produces the new-generation product with unit-cost $c_{n,t}$ by taking the lead time L_n. The manufacturer produces the old-generation product only when the product required for refurbishment service is out of stock. Afterward, the lead time L_o and relatively higher cost (including opportunity cost) $c_{o,t}$ occurs per unit at period t. These forward and reverse flows in the closed-loop system are illustrated in Figure 4.1. When excess inventories of new-generation and old-generation products occur, we assume that the unit inventory holding costs $h_{u,t}$ and $h_{v,t}$, respectively, impose themselves at the end of period t by carrying over to the next period $t+1$. In the case of the unsatisfied demand, the backlog is assumed. We feel that the unit backlog cost of customers who want refurbishment service, p_t, is higher than that of customers who want to buy the new-generation product or join a trade-in program, b_t. Therefore, we assume that the unit penalty cost p_t is relatively larger than b_t in period t for the unsatisfied demands. A supplementary explanation is added to Appendix C.1 to highlight the difference between when the trade-in program was introduced and when it was not. Before developing the mathematical formulation, we made the following assumptions:

Assumption 6. *Customers who want a refurbishment service return end-of-life products to a retailer.*

Assumption 7. *Customers who buy the new-generation product through the trade-in program return the end-of-use products to a retailer.*

Assumption 8. *A remanufacturer receives two types of products from the retailer, which are end-of-use products and end-of-life products.*

Assumption 9. *A manufacturer produces only new-generation products, except when the refurbished product is out of stock.*

Assumption 10. *Lead times and costs for producing the new-generation and old-generation products, and remanufacturing and refurbishing processes are different.*

The quality of the returned product can be classified into several levels that could variegate the lead times and remanufacturing costs. However, we assumed returned

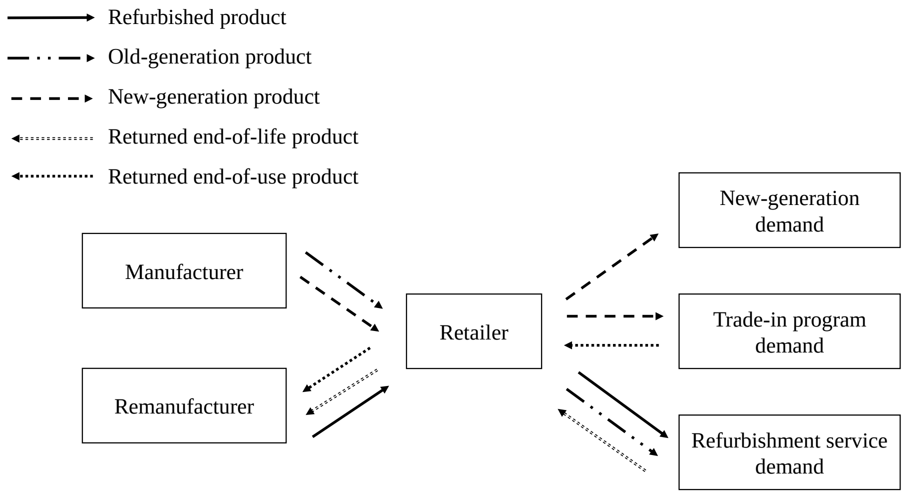


Figure 4.1: Flow of products within the supply chain

products as two levels, including end-of-use and end-of-life products, to focus on the operation of the closed-loop supply chain under the correlated uncertain demands. In the cases of Assumptions 6 and 7, two customer types, who receive a refurbishment service by returning end-of-use products and who participate in the trade-in program by returning end-of-life products, could be considered further. In the former case, however, the retailer will not offer the warranty service to customers who return the products that have been used for a long time but still function properly. Also, if customers are reasonable, they will buy the new-generation product through a trade-in program and not receive the refurbishment service when an additional cost occurs. Therefore, we ruled out the former case in this study. In the latter case, the submission of an end-of-life product to the trade-in program was also ruled out because the retailer would either refuse to honor the return or offer a minimal benefit for it. From a retailer's perspective, even if customers participate in a trade-in program for a small benefit, such a program will not differ much, in terms of profit, from selling new-generation products to those customers. Thus, the latter case was also excluded in this study.

4.3.1 Demand modeling

We assumed that three types of uncertain demands, which are for the new-generation product, refurbishment service, and trade-in program, are correlated. When the release of the new-generation product is successful, customers who use the old-generation product will typically prefer the new-generation product without using the refurbishment service. When the trade-in program is introduced, it can encroach on the refurbishment service. That is, the correlation between the demands of new-generation and refurbishment service shows negative. Also, the correlation between the demands of the trade-in program and refurbishment service features negative. Let $\eta_{n,r}$ and $\eta_{r,t}$ represent the correlations between demands of new-generation product and refurbishment service, and refurbishment service and trade-in program, respectively. Then, the correlation between trade-in program and refurbishment service, $\eta_{t,n}$, can be derived by the $Cauchy - Schwarz\ inequality$ as follows ([72]):

$$\eta_{t,n} \leq \eta_{n,r}\eta_{r,t} + \sqrt{(1-\eta_{n,r}^2)(1-\eta_{r,t}^2)}$$
$$\eta_{t,n} \geq \eta_{n,r}\eta_{r,t} - \sqrt{(1-\eta_{n,r}^2)(1-\eta_{r,t}^2)}$$

From the above inequalities, the range of the correlation between the demands of the trade-in program and new-generation product can be identified. If the inequality $\eta_{n,r}^2 + \eta_{r,t}^2 \geq 1$ holds true, the correlation between the demands of the trade-in program and new-generation product features positive. The three types of demands are correlated with both positive and negative, which makes the demand model complicated in an uncertain environment. That is, the demand modeling method which can capture the correlation is required. Accordingly, we adopted a factor-based demand model as follows:

$$
\begin{aligned}
\tilde{\zeta}_t(\tilde{\mathbf{z}}_t) &\triangleq \zeta_t^0 + \sum_{k=1}^{N} \zeta_t^k \tilde{z}_k \\
\tilde{\phi}_t(\tilde{\mathbf{z}}_t) &\triangleq \phi_t^0 + \sum_{k=1}^{N} \phi_t^k \tilde{z}_k \\
\tilde{\psi}_t(\tilde{\mathbf{z}}_t) &\triangleq \psi_t^0 + \sum_{k=1}^{N} \psi_t^k \tilde{z}_k
\end{aligned} \tag{4.1}
$$

where $1 \leq N_1 \leq N_2 \leq \cdots \leq N_{T-1} = N$ and $\tilde{\mathbf{z}}_k$ are unfolded until $k = 1, \cdots, N_t$.

The factor-based demand model is the stochastic demand model that is affinely dependent on the predefined uncertain factors. Each demand model shares some uncertain factors and thus captures the correlations. Another advantage of the factor-based demand model is that it does not require full information about demand distribution. It can be characterized by only the mean, support, and covariance of the uncertain factor. We describe in detail how the correlation among the three types of demands is incorporated with mean, covariance, and support of the uncertain factor in Section 4.5.1. To sum up, the correlations among the three types of uncertain demands can be captured by adopting the factor-based demand model without full information about the demand distribution.

4.3.2 Decision of the inventory manager

We assume that the inventory manager of the retailer makes the four types of decisions at the beginning of each period, simultaneously. These decisions are (i) the order quantity of the new-generation product, (ii) the order quantity of the old-generation product, (iii) the refurbishing quantity with the end-of-use product, and (iv) the remanufacturing quantity with the end-of-life product. Decision (i) is made to respond to future demands of the new-generation product and trade-in program from the manufacturer. On the other hand, decision (ii) is made to attain the old-generation product from the manufacturer to respond to the refurbishment service when the refurbished product is out of stock. Decisions (iii) and (iv) are made to attain the like-new conditioned product for the refurbishment service from the remanufacturer. The sequence of these decisions by the inventory manager is illustrated in Figure 4.2.

To distinguish the flow of each product from the retailer, we partitioned the product flow with four types; (i) new-generation product, (ii) refurbishment service (old-generation and refurbished products), (iii) refurbishing process (end-of-use product), and (iv) remanufacturing process (end-of-life product). Their flows can be merged, but we partitioned the flow to incorporate the different costs of the product from process to process. The retailer receives the new-generation product from the manufacturer to respond to the demands of the trade-in program and new-generation product. To handle the refurbishment service demand, the retailer acquires like-new conditioned products from a remanufacturer or old-generation products from a manufacturer. The like-new conditioned product should be regarded as two types, which are from end-of-use and end-of-life products because different costs are imposed on

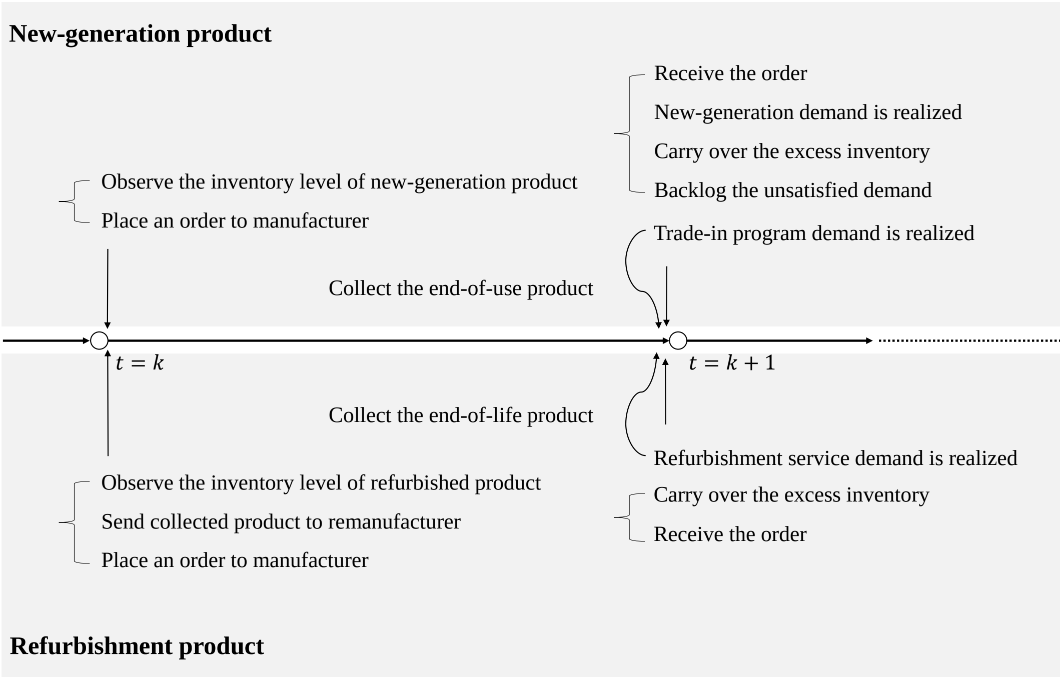


Figure 4.2: Sequence of the decisions made by the inventory manager

them. For the refurbishing process, which transforms end-of-use products to like-new conditioned products, the retailer receives the same quantity of end-of-use products with the demand generated by the trade-in program. Then, the retailer determines how many products will be sent to the remanufacturer or kept in inventory. In the remanufacturing process, which transforms end-of-life products to like-new conditioned products, the retailer receives the same quantity of end-of-life products with the demand generated by refurbishment service. In the same manner as used in the refurbishing process, the retailer determines how many products will be sent to the remanufacturer or kept in inventory. For the mathematical formulation, flow conservation of each inventory process is illustrated in Figure 4.3 and related decision variables are summarized as follows:

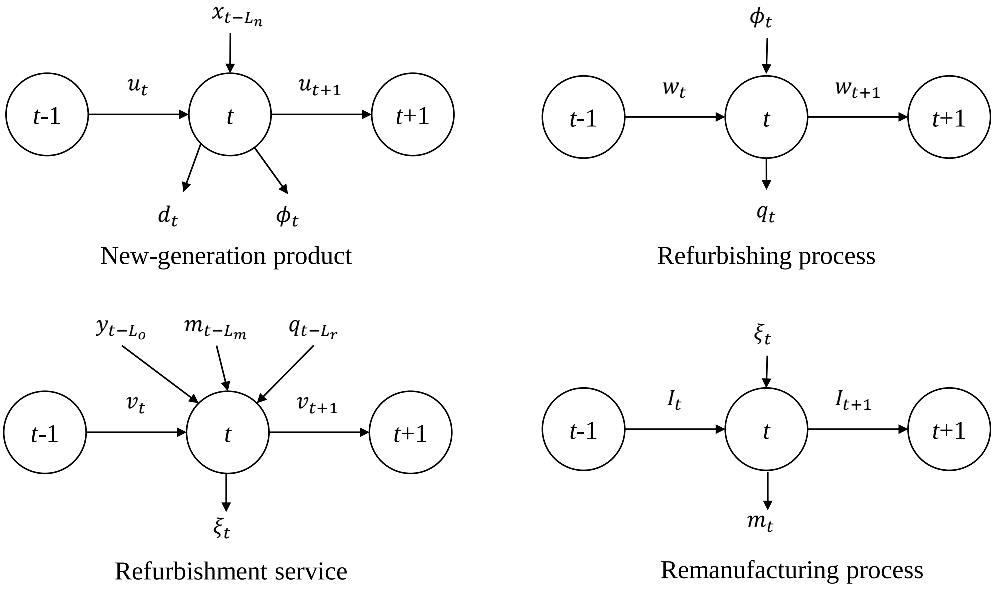


Figure 4.3: Balance equations for the four types of inventories

$x_{n,t}$: Order quantity of new-generation products from a manufacturer at period t

$y_{o,t}$: Order quantity of old-generation products from a manufacturer at period t

q_t : Quantity of end-of-use products sent to a remanufacturer for refurbishing process at period t

m_t : Quantity of end-of-life products sent to a remanufacturer for remanufacturing process at period t

$u_{n,t}$: Inventory level related to new-generation products at period t

$v_{o,t}$: Inventory level related to refurbishment service at period t

w_t : Inventory level related to the refurbishing process at period t

I_t : Inventory level related to the remanufacturing process at period t

4.4 Mathematical formulation

4.4.1 Mathematical formulation of the IMRSTIP under the deterministic demand model

This subsection presents the mathematical formulation under the deterministic demand model. In this model, the inventory manager regards all demands across the entire planning horizon as deterministic values. The notations, ζ_t, ψ_t, and ϕ_t, are denoted to distinguish the deterministic demands from the uncertain demands, $\tilde{\zeta}_t$, $\tilde{\psi}_t$, and $\tilde{\phi}_t$, respectively. The objective of the inventory manager is to minimize the total costs within the entire planning horizon, including purchasing, excess inventory holding, and backlog costs. Under the deterministic demand model, the total costs of the entire planning horizon are represented as follows:

$$\text{Total purchasing costs (TPC)} = \sum_{t\in\mathcal{T}} [c_{n,t}x_{n,t} + c_{o,t}y_{o,t} + c_{r,t}q_t + c_{m,t}m_t] \tag{4.2}$$

$$\text{Total holding costs (THC)} = \sum_{t\in\mathcal{T}} \left[h_{u,t}(u_{n,t+1})^+ + h_{v,t}(v_{o,t+1})^+ + h_{w,t}(w_{t+1})^+ + h_{I,t}(I_{t+1})^+\right]$$

$$\text{Total backlog costs (TBC)} = \sum_{t\in\mathcal{T}} \left[b_t(u_{n,t+1})^- + p_t(v_{o,t+1})^-\right]$$

where $\mathcal{T} \triangleq \{1, \ldots, T-1\}$

Remark 4. *The backlogged inventories related to $\boldsymbol{w}$ and $\boldsymbol{I}$ are not featured in this model. Since the quantities of remanufacturing and refurbishing cannot exceed the returned products through the refurbishment service and trade-in program, backlogged inventories are not presented in TBC. We describe them in detail in Remark 5.*

The inventory manager has to consider the capacities of the manufacturer and remanufacturer. That is, we assume that order quantities of new-generation and old-generation products are limited to the upper bound, C_t, at period t. We also assume that the upper bound, U_t, is limited for remanufacturing and refurbishing processes at period t. In addition, remanufacturing quantity cannot exceed the available end-of-use products. In a similar manner, refurbishing quantity cannot exceed the available end-of-life products. By taking account of the related capacity constraints and flow conservation constraints, which are illustrated in Figure 4.3, we developed the linear program (4.3) as follows:

$$
\begin{aligned}
\min \quad & \mathrm{TPC} + \mathrm{THC} + \mathrm{TBC} & & \\
\text{s.t.} \quad & u_{n,t+1} = u_{n,t} + x_{n,t-L_n} - \zeta_t - \phi_t & & t \in \mathcal{T}; \\
& v_{o,t+1} = v_{o,t} + y_{o,t-L_o} + m_{t-L_m} + q_{t-L_r} - \psi_t & & t \in \mathcal{T}; \\
& w_{t+1} = w_t + \phi_t - q_t & & t \in \mathcal{T}; \\
& I_{t+1} = I_t + \psi_t - m_t & & t \in \mathcal{T}; \\
& x_{n,t} + y_{o,t} \leq C_t & & t \in \mathcal{T}; \\
& q_t + m_t \leq U_t & & t \in \mathcal{T}; \\
& q_t \leq w_t + \phi_t & & t \in \mathcal{T}; \\
& m_t \leq I_t + \psi_t & & t \in \mathcal{T}; \\
& x_{n,t}, y_{o,t}, q_t, \text{ and } m_t \geq 0 & & t \in \mathcal{T};
\end{aligned}
\tag{4.3}
$$

Remark 5. *Constraints related to the quantities of refurbishing and remanufacturing can be reformulated as the non-negative constraints of inventory levels w_{t+1} and I_{t+1}, respectively, as follows:*

$$q_t \leq w_t + \phi_t \iff w_t + \phi_t - q_t \geq 0 \iff w_{t+1} \geq 0 \qquad t \in \mathcal{T}; \tag{4.4}$$

$$m_t \leq I_t + \psi_t \iff I_t + \psi_t - m_t \geq 0 \iff I_{t+1} \geq 0 \qquad t \in \mathcal{T};$$

We do not include the fixed cost for purchasing the product from the manufacturer or remanufacturer. Also, we relax the decision variables as the real variables, which could be considered as integer variables. Although the fixed cost or integer variable can be considered, we formulated a mathematical model as a linear program to retain tractability for the distributionally robust optimization approach, as several previous studies did ([3, 60, 88, 95]).

4.4.2 Mathematical formulation of the IMRSTIP under the stochastic demand model

For the inventory manager, it is challenging to predict future demand exactly as a nominal value. When the realized demand differs from the predicted nominal value, this can occur at an enormous cost. To incorporate the uncertain demand while capturing the correlation among three types of demands, we adopt the factor-based demand model. Due to uncertain factors, the objective function is expressed as an expectation form $\mathbb{E}(\cdot)$. Accordingly, the objective function is to minimize the expectation of total costs during the entire planning horizon. The expectation of total purchasing, inventory holding, and backlog costs under the stochastic demand, TPC-S, THC-S, TBC-S, respectively, can be represented as follows:

$$\text{TPC-S} = \mathbb{E}\left[\sum_{t\in\mathcal{T}} \left(c_{n,t}x_{n,t}(\tilde{\mathbf{z}}_{t-1}) + c_{o,t}y_{o,t}(\tilde{\mathbf{z}}_{t-1}) + c_{r,t}q_t(\tilde{\mathbf{z}}_{t-1}) + c_{m,t}m_t(\tilde{\mathbf{z}}_{t-1})\right)\right] \tag{4.5}$$

$$\text{THC-S} = \mathbb{E}\left[\sum_{t\in\mathcal{T}} \left(h_{u,t}(u_{n,t+1}(\tilde{\mathbf{z}}_t))^+ + h_{v,t}(v_{o,t+1}(\tilde{\mathbf{z}}_t))^+ + h_{w,t}(w_{t+1}(\tilde{\mathbf{z}}_t))^+ + h_{I,t}(I_{t+1}(\tilde{\mathbf{z}}_t))^+\right)\right]$$

$$\text{TBC-S} = \mathbb{E}\left[\sum_{t\in\mathcal{T}} \left(b_t(u_{n,t+1}(\tilde{\mathbf{z}}_t))^- + p_t(v_{o,t+1}(\tilde{\mathbf{z}}_t))^-\right)\right]$$

Based on the expected total costs presented in (4.5), the multistage stochastic optimization model can be developed as follows:

$$
\begin{aligned}
\min \quad & \text{TPC-S} + \text{THC-S} + \text{TBC-S} && (4.6)\\
\text{s.t.} \quad & u_{n,t+1}(\tilde{\mathbf{z}}_t) = u_{n,t}(\tilde{\mathbf{z}}_{t-1}) + x_{n,t-L_n}(\tilde{\mathbf{z}}_{t-L_n-1}) - \zeta_t(\tilde{\mathbf{z}}_t) - \phi_t(\tilde{\mathbf{z}}_t) && t\in\mathcal{T};\\
& v_{o,t+1}(\tilde{\mathbf{z}}_t) = v_{o,t}(\tilde{\mathbf{z}}_{t-1}) + y_{o,t-L_o}(\tilde{\mathbf{z}}_{t-L_o-1}) + m_{t-L_m}(\tilde{\mathbf{z}}_{t-L_m-1}) + q_{t-L_r}(\tilde{\mathbf{z}}_{t-L_r-1}) - \psi_t(\tilde{\mathbf{z}}_t) && t\in\mathcal{T};\\
& w_{t+1}(\tilde{\mathbf{z}}_t) = w_t(\tilde{\mathbf{z}}_{t-1}) + \phi_t(\tilde{\mathbf{z}}_t) - q_t(\tilde{\mathbf{z}}_{t-1}) && t\in\mathcal{T};\\
& I_{t+1}(\tilde{\mathbf{z}}_t) = I_t(\tilde{\mathbf{z}}_{t-1}) + \psi_t(\tilde{\mathbf{z}}_t) - m_t(\tilde{\mathbf{z}}_{t-1}) && t\in\mathcal{T};\\
& x_{n,t}(\tilde{\mathbf{z}}_{t-1}) + y_{o,t}(\tilde{\mathbf{z}}_{t-1}) \le C_t && t\in\mathcal{T};\\
& q_t(\tilde{\mathbf{z}}_{t-1}) + m_t(\tilde{\mathbf{z}}_{t-1}) \le U_t && t\in\mathcal{T};\\
& q_t(\tilde{\mathbf{z}}_{t-1}) \le w_t(\tilde{\mathbf{z}}_t) + \phi_t(\tilde{\mathbf{z}}_t) && t\in\mathcal{T};\\
& m_t(\tilde{\mathbf{z}}_{t-1}) \le I_t(\tilde{\mathbf{z}}_t) + \psi_t(\tilde{\mathbf{z}}_t) && t\in\mathcal{T};\\
& x_{n,t}(\tilde{\mathbf{z}}_{t-1}), y_{o,t}(\tilde{\mathbf{z}}_{t-1}), q_t(\tilde{\mathbf{z}}_{t-1}), \text{ and } m_t(\tilde{\mathbf{z}}_{t-1}) \ge 0 && t\in\mathcal{T};
\end{aligned}
$$

4.4.3 Distributionally robust optimization approach for the IMRSTIP

In general, probability distributions of the random variables are assumed to be known or estimated in the stochastic optimization model ([93]). In other words, the inventory manager should decide on the order quantity based on the stochastic demand with full information. In practice, however, obtaining full information about uncertain factors is difficult. Even if the distribution is estimated exactly, in general, evaluating the expected value in the multistage decision process is computationally intractable ([90, 92, 91]). Instead, we assumed that the first and second moments and support of the uncertain factors are available for the inventory manager. By approximating the upper bound of the multistage stochastic optimization model (4.6), we could derive a robust counterpart that is tractable. That is, minimizing the upper bound was considered rather than directly minimizing the expected value. To approximate the multistage stochastic optimization model to the tractable deterministic model, the following assumption is required for the uncertain factors $\tilde{\mathbf{z}}$.

Assumption B. (Chen and Sim [26]) *Uncertain factors $\tilde{\mathbf{z}}$ are zero-mean random variables, $\mathbb{E}[\tilde{z}_k] = 0, \forall k \in 1, \ldots, N$, with the positive definite covariance matrix $\mathbf{\Sigma}$. The uncertain factors are defined on the support set $\boldsymbol{W}$ which is a second-order conic representable set, such as intervals, polyhedrons, or ellipsoids.*

Linear decision rule

We adopted the *linear decision rule* (LDR) to handle the multistage stochastic optimization model. By restricting decision variables as affinely dependent on the uncertain factors, the decision can be delayed by observing the realization of part of the uncertain factors. Related decision variables with the LDR are expressed as

follows:

$$
\begin{aligned}
{x_{n,t}}^{\mathrm{LDR}}(\tilde{\boldsymbol{z}}) &= x_{n,t}^{0} + \sum_{k=1}^{N} x_{n,t}^{k} \tilde{z}_k \\
{y_{o,t}}^{\mathrm{LDR}}(\tilde{\boldsymbol{z}}) &= y_{o,t}^{0} + \sum_{k=1}^{N} y_{o,t}^{k} \tilde{z}_k \\
{q_t}^{\mathrm{LDR}}(\tilde{\boldsymbol{z}}) &= q_t^{0} + \sum_{k=1}^{N} q_t^{k} \tilde{z}_k \\
{m_t}^{\mathrm{LDR}}(\tilde{\boldsymbol{z}}) &= m_t^{0} + \sum_{k=1}^{N} m_t^{k} \tilde{z}_k
\end{aligned}
\tag{4.7}
$$

Because the decision is based on the realized uncertain factors, which is referred to as the *non anticipative* property, uncertain factors are restricted that are unavailable from period t. This property can be incorporated by adding non-anticipative constraints as follows:

$$
\begin{aligned}
& x_{n,t}^{k} = 0, \quad \forall k \geq N_{t-1} + 1 \\
& y_{o,t}^{k} = 0, \quad \forall k \geq N_{t-1} + 1 \\
& q_t^{k} = 0, \quad \forall k \geq N_{t-1} + 1 \\
& m_t^{k} = 0, \quad \forall k \geq N_{t-1} + 1
\end{aligned}
\tag{4.8}
$$

Hence, the order decision is based on the observed information available at the beginning of each period t.

Remark 6. *The decision variables relevant to inventory levels, $u_{n,t+1}$, v_{t+1}, w_{t+1}, and I_{t+1}, are also affinely dependent on uncertain factors, $\tilde{\boldsymbol{z}}$, as follows:*

$$u_{n,t}(\tilde{\boldsymbol{z}}) = u_{n,t}^0 + \sum_{k=1}^{N} u_{n,t}^k \tilde{z}_k$$
$$v_{o,t}(\tilde{\boldsymbol{z}}) = v_{o,t}^0 + \sum_{k=1}^{N} v_{o,t}^k \tilde{z}_k$$
$$w_t(\tilde{\boldsymbol{z}}) = w_t^0 + \sum_{k=1}^{N} w_t^k \tilde{z}_k$$
$$I_t(\tilde{\boldsymbol{z}}) = I_t^0 + \sum_{k=1}^{N} I_t^k \tilde{z}_k$$

By deriving the balance equation for the inventory level with the closed-form expression, it can be easily discerned that decision variables indicating inventory levels are also affine function of the uncertain factors:

$$\begin{aligned} u_{n,t+1}(\tilde{\boldsymbol{z}}) &= u_1^0 + \sum_{i=1}^{t} x_{n,i}(\tilde{\boldsymbol{z}}) - \sum_{i=1}^{t} \zeta_i(\tilde{\boldsymbol{z}}) \\ &= u_1^0 + \sum_{i=1}^{t} \left((x_{n,i}^0 - \zeta_i^0) + \sum_{k=1}^{N} (x_{n,i}^k - \zeta_i^k) \tilde{z}_k \right) \\ &= u_{n,t}^0 + \sum_{k=1}^{N} u_{n,t}^k \tilde{z}_k \end{aligned} \tag{4.9}$$

The remaining three inventory levels can also be derived in the same manner. From the non-anticipative constraints in (4.8), inventory levels also feature the non-anticipative property seen in the second equality in (4.9). As a result, decision variables related to the inventory level also feature the non-anticipative property as

follows:

$$
\begin{aligned}
&u_{n,t+1}^{k}=0, \quad \forall k \geq N_t+1 \\
&v_{o,t+1}^{k}=0, \quad \forall k \geq N_t+1 \\
&w_{t+1}^{k}=0, \quad \forall k \geq N_t+1 \\
&I_{t+1}^{k}=0, \quad \forall k \geq N_t+1
\end{aligned}
\tag{4.10}
$$

Upper bound of the expected positive parts

Instead of directly minimizing the expectation of the multistage stochastic optimization model, we primarily focused on minimizing the approximated upper bound of the expected value. The upper bound of the expected value, which is the worst-case expected cost, can be estimated by utilizing the distributionally robust bound and LDR. The expected purchasing cost can be obtained directly but expected holding and backlog costs are approximated to the upper bounds. For the purchasing cost of new-generation products, the expected value can be obtained by the following process:

$$
\begin{aligned}
&\mathbb{E}\left(c_{n,t}(x_{n,t}^{0}+\sum_{k=1}^{N} x_{n,t}^{k}\tilde{z}_k)\right) \\
&=c_{n,t}\mathbb{E}\left(x_{n,t}^{0}\right)+c_{n,t}\mathbb{E}\left(\sum_{k=1}^{N} x_{n,t}^{k}\tilde{z}_k\right) \\
&=c_{n,t}x_{n,t}^{0} \\
&\because \ \mathbb{E}[\tilde{z}_k]=0
\end{aligned}
\tag{4.11}
$$

In the same manner, total purchasing costs based on the LDR can be expressed as follows:

$$\text{TPC-R} = \sum_{t\in\mathcal{T}} \left(c_{n,t}x^0_{n,t} + c_{o,t}y^0_{o,t} + c_{r,t}q^0_t + c_{m,t}m^0_t\right)$$

By adopting the work of Chen and Sim [26], we derived three upper bounds of expected positive parts related to excess inventories. The first bound is presented in (4.12) as follows:

$$\begin{aligned}
&\mathbb{E}\left((u^0_{n,t+1} + \sum_{k=1}^{N} u^k_{n,t+1}\tilde{z}_k)^+\right)\\
&\leq \left(u^0_{n,t+1} + \max_{\tilde{\mathbf{z}}\in\mathbf{W}} \sum_{k=1}^{N} u^k_{n,t+1}\tilde{z}_k\right)^+\\
&= \pi^1(u^0_{n,t+1}, \mathbf{u}_{t+1})
\end{aligned} \tag{4.12}$$

The second bound can be derived by using the equality $a^+ = a + (-a)^+$

$$\begin{aligned}
&\mathbb{E}\left((u^0_{n,t+1} + \sum_{k=1}^{N} u^k_{n,t+1}\tilde{z}_k)^+\right)\\
&= u^0_{n,t+1} + \mathbb{E}\left((-u^0_{n,t+1} - \sum_{k=1}^{N} u^k_{n,t+1}\tilde{z}_k)^+\right)\\
&\leq u^0_{n,t+1} + \left(-u^0_{n,t+1} + \max_{\tilde{\mathbf{z}}\in\mathbf{W}} \sum_{k=1}^{N} (-u^k_{n,t+1})\tilde{z}_k\right)^+\\
&= \pi^2(u^0_{n,t+1}, \mathbf{u}_{t+1})
\end{aligned} \tag{4.13}$$

The third bound can be derived by using the equality $a^+ = (a+|a|)/2$ and Jensen's inequality as follows:

$$
\begin{aligned}
&\mathbb{E}\left((u^0_{n,t+1} + \sum_{k=1}^{N} u^k_{n,t+1}\tilde{z}_k)^+\right) \\
&= \frac{1}{2}\mathbb{E}\left(u^0_{n,t+1} + \sum_{k=1}^{N} u^k_{n,t+1}\tilde{z}_k\right) + \frac{1}{2}\mathbb{E}\Big|u^0_{n,t+1} + \textstyle\sum_{k=1}^{N} u^k_{n,t+1}\tilde{z}_k\Big| \\
&= \frac{1}{2}u^0_{n,t+1} + \frac{1}{2}\mathbb{E}\Big|u^0_{n,t+1} + \textstyle\sum_{k=1}^{N} u^k_{n,t+1}\tilde{z}_k\Big| \qquad (4.14) \\
&\leq \frac{1}{2}u^0_{n,t+1} + \frac{1}{2}\sqrt{\mathbb{E}[((u^0_{n,t+1} + \sum_{k=1}^{N} u^k_{n,t+1}\tilde{z}_k)^+)^2]} \qquad \text{(by Jensen's inequality)} \\
&= \tfrac{1}{2}u^0_{n,t+1} + \tfrac{1}{2}\sqrt{(u^0_{n,t+1})^2 + \mathbb{E}[\left(\textstyle\sum_{k=1}^{N} u^k_{n,t+1}\tilde{z}_k\right)^2]} \\
&= \pi^3(u^0_{n,t+1}, \mathbf{u}_{t+1})
\end{aligned}
$$

Theorem 2. *The upper bound of the expected positive part* $\mathbb{E}((u^0_{n,t+1} + \mathbf{u}'_{t+1}\tilde{\mathbf{z}})^+)$, *which is represented by* $\pi(u^0_{n,t+1}, \mathbf{u}_{t+1})$, *can be obtained by minimizing the three bounds,* $\pi^1(u^0_{n,t+1}, \mathbf{u}_{t+1})$, $\pi^2(u^0_{n,t+1}, \mathbf{u}_{t+1})$, *and* $\pi^3(u^0_{n,t+1}, \mathbf{u}_{t+1})$ *as follows:*

$$
\begin{aligned}
\pi(u^0_{n,t+1}, \mathbf{u}_{t+1}) \triangleq \min &\sum_{i=1}^{3} \pi^i(u^0_{i,t+1}, \mathbf{u}_{i,t+1}) \\
\text{s.t.} &\sum_{i=1}^{3} u^0_{i,t+1} = u^0_{n,t+1} \qquad (4.15) \\
&\sum_{i=1}^{3} \mathbf{u}_{i,t+1} = \mathbf{u}_{t+1}
\end{aligned}
$$

The optimization problem (4.15) for every t-th period ($t \in \mathcal{T}$) can be expressed as the epigraph form by combining the inequalities (4.12), (4.13), and (4.14) as follows ([26]):

$$
\begin{aligned}
\pi(u^0_{n,t+1}, \mathbf{u}_{t+1}) = \min \quad & r_{1,t+1} + r_{2,t+1} + r_{3,t+1} \\
\text{s.t.} \quad & u^0_{1,t+1} + \max_{\tilde{\mathbf{z}} \in \mathbf{W}} \tilde{\mathbf{z}}' \mathbf{u}_{1,t+1} \le r_{1,t+1} \\
& r_{1,t+1} \ge 0 \\
& \max_{\tilde{\mathbf{z}} \in \mathbf{W}} \tilde{\mathbf{z}}'(-\mathbf{u}_{2,t+1}) \le r_{2,t+1} \\
& u^0_{2,t+1} \le r_{2,t+1} \\
& \frac{1}{2} u^0_{3,t+1} + \frac{1}{2} \left| u^0_{3,t+1}, \Sigma^{1/2} \mathbf{u}_{3,t+1} \right|_2 \le r_{3,t+1} \\
& u^0_{1,t+1} + u^0_{2,t+1} + u^0_{3,t+1} = u^0_{n,t+1} \\
& \mathbf{u}_{1,t+1} + \mathbf{u}_{2,t+1} + \mathbf{u}_{3,t+1} = \mathbf{u}_{t+1} \\
& r_{i,t+1}, u^0_{i,t+1} \in \mathcal{R}, \quad \mathbf{u}_{i,t+1} \in \mathcal{R}^N, \quad i = 1, \ldots, 3
\end{aligned}
\tag{4.16}
$$

Remark 7. *Due to the Assumption B, the optimization problem* (4.16) *is tractable if the robust counterparts of the inner optimizations in the first and third constraints are appropriately defined. If the uncertain factors are not defined on the support set* $\boldsymbol{W}$*, the optimization problem* (4.16) *becomes intractable, which leads to a robust optimization model also being intractable.*

To sum up, we approximated the objective function of the multistage stochastic optimization model, which was expressed as the expectation form. The approximated upper bound of the objective function features the second-order cone program because the quadratic constraint remains on the third upper bound of the expected positive part (4.14). According to the Ben-Tal et al. [10], the robust counterpart retains tractability if the ellipsoidal uncertainty set is considered in a linear program or second-order cone program (which is also known as a conic quadratic program). For the cases of the interval or polyhedron sets, these sets do not affect the complexity of the problem. In this study, under Assumption B, the optimization model (4.16) can retain the property of the second-order cone program if the robust counterpart is well defined by handling the remaining uncertain factors properly. Thus, the problem can be solved by the interior point method with the commercial optimization solver. If the support set $\boldsymbol{W}$ is not a second-order conic representable set, such as an intersection of ellipsoids, the robust counterpart of the problem becomes NP-hard which leads to the robust optimization model being intractable ([10]).

By solving the optimization problem (4.16), a tighter upper bound can be achieved than can be achieved for each bound as presented in inequality (4.17).

$$\mathbb{E}\left((u_{n,t+1}^0 + \sum_{k=1}^{N} u_{n,t+1}^k \tilde{z}_k)^+\right) \leq \pi(u_{n,t+1}^0, \mathbf{u}_{t+1}) \leq \min_{i=1,2,3} \pi^i(u_{n,t+1}^0, \mathbf{u}_{t+1}) \tag{4.17}$$

Remark 8. *A reasonable upper bound can be approximated without utilizing directional deviations, which are the forward and backward deviations* ([88]).

Chen and Sim [26] and See and Sim [88] used fourth and fifth bounds which are derived from the forward and backward deviations, respectively, in the optimization model (4.16). However, a close bound can be achieved even if the directional deviations are set to ∞. If the information of the deviations is unavailable, it could be omitted by forcing the fourth and fifth bounds to be redundant.

Using the same process, we can derive the upper bound of the total inventory holding costs for the entire planning horizon as follows:

$$\text{THC-R} = \sum_{t\in\mathcal{T}} (h_{u,t}\pi(u^0_{n,t+1}, \mathbf{u}_{t+1}) + h_{v,t}\pi(v^0_{o,t+1}, \mathbf{v}_{t+1}) + h_{w,t}\pi(w^0_{t+1}, \mathbf{w}_{t+1})$$
$$+ h_{I,t}\pi(I^0_{t+1}, \mathbf{I}_{t+1}))$$

The upper bound of expected backlogged inventory can also be derived as follows:

$$\mathbb{E}\left((u^0_{n,t+1} + \sum_{k=1}^{N} u^k_{n,t+1}\tilde{z}_k)^-\right) \le \pi(-u^0_{n,t+1}, -\mathbf{u}_{t+1}) \le \min_{i=1,2,3} \pi^i(-u^0_{n,t+1}, -\mathbf{u}_{t+1}) \tag{4.18}$$

The total backlog cost can be approximated based on (4.18) as follows:

$$\text{TBC-R} = \sum_{t\in\mathcal{T}} \left(b_t\pi(-u^0_{n,t+1}, -\mathbf{u}_{t+1}) + p_t\pi(-v^0_{o,t+1}, -\mathbf{v}_{t+1})\right)$$

Consequently, the upper bound of the total costs on the entire planning horizon, including TPC-R, THC-R, and TBC-R, are derived based on the LDR.

Robust counterpart

Based on the LDR formulations represented in (4.7), the approximated upper bound of the multistage stochastic optimization model (4.6) are derived as follows:

$$
\begin{aligned}
\min \quad & \text{TPC-R} + \text{THC-R} + \text{TBC-R} \\
\text{s.t.} \quad & u^k_{n,t+1} = u^k_{n,t} + x^k_{n,t-L_n} - \zeta^k_t - \phi^k_t && t \in \mathcal{T}; k \in 0 \dots N \\
& v^k_{o,t+1} = v^k_{o,t} + y^k_{o,t-L_o} + m^k_{t-L_m} + q^k_{t-L_r} - \psi^k_t && t \in \mathcal{T}; k \in 0 \dots N \\
& w^k_{t+1} = w^k_t + \phi^k_t - q^k_t && t \in \mathcal{T}; k \in 0 \dots N \\
& I^k_{t+1} = I^k_t + \psi^k_t - m^k_t && t \in \mathcal{T}; k \in 0 \dots N \\
& x^k_{n,t},\ y^k_{o,t},\ q^k_t,\ m^k_t = 0 && k \geq N_{t-1} + 1;\ t \in \mathcal{T}; \\
& u^k_{n,t+1},\ v^k_{o,t+1},\ w^k_{t+1},\ I^k_{t+1} = 0 && k \geq N_t + 1;\ t \in \mathcal{T}; \\
& x^0_{n,t} + \mathbf{x_t}'\tilde{\mathbf{z}} + y^0_{o,t} + \mathbf{y_t}'\tilde{\mathbf{z}} \leq C_t && \tilde{\mathbf{z}} \in \boldsymbol{W},\ t \in \mathcal{T}; \qquad (4.19) \\
& q^0_t + \mathbf{q_t}'\tilde{\mathbf{z}} + m^0_t + \mathbf{m_t}'\tilde{\mathbf{z}} \leq U_t && \tilde{\mathbf{z}} \in \boldsymbol{W},\ t \in \mathcal{T}; \\
& q^0_t + \mathbf{q_t}'\tilde{\mathbf{z}} \leq w^0_t + \mathbf{w_t}'\tilde{\mathbf{z}} + \phi^0_t + \boldsymbol{\phi_t}'\tilde{\boldsymbol{z}} && \tilde{\mathbf{z}} \in \boldsymbol{W},\ t \in \mathcal{T}; \\
& m^0_t + \mathbf{m_t}'\tilde{\mathbf{z}} \leq I^0_t + \mathbf{I_t}'\tilde{\mathbf{z}} + \psi^0_t + \boldsymbol{\xi_t}'\tilde{\boldsymbol{z}} && \tilde{\mathbf{z}} \in \boldsymbol{W},\ t \in \mathcal{T}; \\
& x^0_{n,t} + \mathbf{x_t}'\tilde{\mathbf{z}} \geq 0 && \tilde{\mathbf{z}} \in \boldsymbol{W},\ t \in \mathcal{T}; \\
& y^0_{o,t} + \mathbf{y_t}'\tilde{\mathbf{z}} \geq 0 && \tilde{\mathbf{z}} \in \boldsymbol{W},\ t \in \mathcal{T}; \\
& q^0_t + \mathbf{q_t}'\tilde{\mathbf{z}} \geq 0 && \tilde{\mathbf{z}} \in \boldsymbol{W},\ t \in \mathcal{T}; \\
& m^0_t + \mathbf{m_t}'\tilde{\mathbf{z}} \geq 0 && \tilde{\mathbf{z}} \in \boldsymbol{W},\ t \in \mathcal{T};
\end{aligned}
$$

Constraints related to the inventory balance equation are derived to the linear equations. However, the constraint related to the order capacity for the manufacturer,

C_t, remains the uncertain factors $\tilde{\mathbf{z}} \in \boldsymbol{W}$. Since the uncertain factors are assumed to be bounded and defined on the support set $\boldsymbol{W}$, the constraints can be reformulated to the robust counterpart by defining the inner optimization as follows:

$$\begin{aligned} & x^0_{n,t} + \mathbf{x_{n,t}}'\tilde{\mathbf{z}} + y^0_{o,t} + \mathbf{y_{o,t}}'\tilde{\mathbf{z}} \leq C_t, \quad \tilde{\mathbf{z}} \in \boldsymbol{W} \\ \iff & x^0_{n,t} + y^0_{o,t} + \max_{\tilde{\mathbf{z}} \in \boldsymbol{W}} \textstyle\sum_{k=1}^{N} (x^k_{n,t} + y^k_{o,t})\tilde{z}_k \leq C_t \end{aligned} \tag{4.20}$$

In the case of the capacity constraints for the remanufacturer, U_t, we followed the same procedure as shown in (4.20) and obtained the robust counterpart as follows:

$$\begin{aligned} & q^0_t + \mathbf{q_t}'\tilde{\mathbf{z}} + m^0_t + \mathbf{m_t}'\tilde{\mathbf{z}} \leq U_t, \quad \tilde{\mathbf{z}} \in \boldsymbol{W} \\ \iff & q^0_t + m^0_t + \max_{\tilde{\mathbf{z}} \in \boldsymbol{W}} \textstyle\sum_{k=1}^{N} (q^k_t + m^k_t)\tilde{z}_k \leq U_t \end{aligned} \tag{4.21}$$

Recall that the constraints related to the refurbishing and remanufacturing quantities can be reformulated as (4.4). Consequently, we derived the robust counterpart of the constraints related to the refurbishing quantity for every period $t \in \mathcal{T}$ as follows:

$$\begin{aligned} & q_t(\tilde{z}_{t-1}) \leq w_t(\tilde{z}_t) + \phi_t(\tilde{z}_t) \iff w_{t+1}(\tilde{z}_t) \geq 0 \\ \iff & w^0_{t+1} - \max_{\tilde{\mathbf{z}} \in \boldsymbol{W}} \textstyle\sum_{k=1}^{N} w^k_{t+1}\tilde{z}_k \geq 0 \end{aligned} \tag{4.22}$$

Constraints representing the capacity of the remanufacturing quantity can be reformulated as a robust counterpart with the same process used in (4.22) as follows:

$$\begin{aligned} m_t(\tilde{z}_{t-1}) \leq I_t(\tilde{z}_t) + \psi_t(\tilde{z}_t) \iff I_{t+1}(\tilde{z}_t) \geq 0 \\ \iff I^0_{t+1} - \max_{\tilde{\mathbf{z}} \in \boldsymbol{W}} \textstyle\sum_{k=1}^N I^k_{t+1} \tilde{z}_k \geq 0 \end{aligned} \tag{4.23}$$

We also developed robust counterparts for decision variables $x_{n,t}(\tilde{z}_{t-1})$, $y_{o,t}(\tilde{z}_{t-1})$, $q_t(\tilde{z}_{t-1})$, and $m_t(\tilde{z}_{t-1})$ which have non-negative conditions for every period $t \in \mathcal{T}$.

$$\begin{cases} x^0_{n,t} + \mathbf{x_t'}\tilde{\mathbf{z}} \geq 0 & \tilde{\mathbf{z}} \in \boldsymbol{W}; \\ y^0_{o,t} + \mathbf{y_t'}\tilde{\mathbf{z}} \geq 0 & \tilde{\mathbf{z}} \in \boldsymbol{W}; \\ q^0_t + \mathbf{q_t'}\tilde{\mathbf{z}} \geq 0 & \tilde{\mathbf{z}} \in \boldsymbol{W}; \\ m^0_t + \mathbf{m_t'}\tilde{\mathbf{z}} \geq 0 & \tilde{\mathbf{z}} \in \boldsymbol{W}; \end{cases} \iff \begin{cases} x^0_{n,t} - \max_{\tilde{\mathbf{z}} \in \boldsymbol{W}} \sum_{k=1}^N x^k_{n,t} \tilde{z}_k \geq 0 \\ y^0_{o,t} - \max_{\tilde{\mathbf{z}} \in \boldsymbol{W}} \sum_{k=1}^N y^k_{o,t} \tilde{z}_k \geq 0 \\ q^0_t - \max_{\tilde{\mathbf{z}} \in \boldsymbol{W}} \sum_{k=1}^N q^k_t \tilde{z}_k \geq 0 \\ m^0_t - \max_{\tilde{\mathbf{z}} \in \boldsymbol{W}} \sum_{k=1}^N m^k_t \tilde{z}_k \geq 0 \end{cases} \tag{4.24}$$

Under Assumption B, Constraints (4.20) - (4.24) can be transformed to a tractable robust counterpart by reformulating the inner optimization problem properly. Tractability of the inner optimization problem could be handled in the same manner as in Remark 7. In the case of a robust counterpart, it depends on what types of support sets $\boldsymbol{W}$ are defined. For example, if uncertain factors $\tilde{z}_k$ are distributed on a symmetric interval set $\boldsymbol{W} = [-z, z]$ in the constraint $x^0_{n,t} - \max_{\tilde{\mathbf{z}} \in \boldsymbol{W}} \sum_{k=1}^N x^k_{n,t} \tilde{z}_k \geq 0$, this could be handled by introducing the absolute value as $x^0_{n,t} - \sum_{k=1}^N \mid x^k_{n,t} \mid z \geq 0$. In this manner, all uncertain factors remaining in the optimization model (4.19) could be handled. Finally, the robust counterpart of the IMRSTIP (RIMRSTIP) becomes the deterministic second-order cone optimization model.

Theorem 3. *The objective value of the robust counterpart of the optimization model* (4.19) *is greater than or equal to that of the multistage stochastic optimization model* (4.6).

Theorem 3 can be easily derived by combining the inequalities in (4.11), (4.17), and (4.18). Although the robust counterpart of (4.19) provides the objective value that is worse than that obtained from the stochastic optimization problem (4.5), it features the deterministic second-order cone optimization problem that has the virtue of computational tractability. Performance gaps with the multistage stochastic optimization model are provided in Section 4.5.

4.5 Computational experiments

We conducted the computational experiments to validate the performance of the RIMRSTP. Also, we performed experiments to derive managerial insights. In addition to the research questions (i), (ii), and (iii) in Section 4.1, research questions related to the model validation and insights are raised as follows:

- Does the robust model retain tractability until a modest data size?
- What is the loss of the objective value during approximation to the robust counterpart?
- To what extent does the robust model protect against the realized uncertain factors?
- What is the aspect of the inventory operation when the correlations among the three types of demands exist or do not?

All computational experiments were performed by the optimization solver FICO XPRESS-IVE version 7.2 with an Intel® Core™ i5-7400 CPU @ 3.0 GHz. Before reporting the results of the experiments, we provide a small example of the demand process and computational results in Section 4.5.1.

4.5.1 Demand process

We begin by explaining the demand generation. The factor-based demand model, which we adopted from See and Sim [88], involves the constant term and coefficient of the uncertain factors. The factor-based demand model is modeled as the affine function of the uncertain factor by estimating the related parameters. We utilized the time series analysis method to predict future demand and induced it as a factor-based demand model.

ARMA model

The *autoregressive moving − average* (ARMA) model is one of the stochastic processes that predict future points in the time series. It is a generalization model of the autoregressive and moving average models, which are expressed with previous observations of time series data and past error terms, respectively. By estimating future demand with past data from the ARMA (p, q) process, the inventory manager can obtain the future demand as the affine function of the uncertain factors as follows ([88]):

$$\zeta_t(\tilde{\mathbf{z}}_t) \simeq \sum_{k=1}^{p} \alpha_j \zeta_{t-k}(\tilde{\boldsymbol{z}}) + \tilde{z}_t + \sum_{k=1}^{\min(q,t-1)} \beta_j \tilde{z}_{t-j} \tag{4.25}$$

The ARMA model projects the future values of a series based entirely on its inertia. In other words, there exists a limit to capture demand for other products that can affect each other. Thus, we adopted a model that can incorporate the correlation or dependency of different demands.

VARMA model

To capture the dependencies among the three types of demands, we adopted the *vector autoregressive moving−average* (VARMA) model. It is one of the stochastic processes characterizing the dependencies of the multivariate time series. By allowing the error terms to be autocorrelated on the *vector autoregressive* (VAR) model, the

VARMA (p, q), which is a generalized model, can be developed. Unlike the univariate ARMA model, the VARMA model has more than one time-dependent variable. Each variable depends not only on its past values but also has some dependency on other variables. Denote by $\mathbf{D}_t \triangleq \left(\tilde{d}_t, \tilde{\phi}_t, \tilde{\xi}_t\right)$ as a vector of three types of demands at period t. Let $\mathbf{D}_t^0$ denote the vector of constant terms $\left(d_t^0, \phi_t^0, \xi_t^0\right)$. For simplicity, let VARMA (p, q) as VAR(1) which can be representable when the stochastic process is stable ([64]). Then, we can develop the demand model as follows:

$$\mathbf{D}_t = \mathbf{D}_t^0 + \Psi_{t-1}\mathbf{D}_{t-1} + \boldsymbol{\epsilon}_t \tag{4.26}$$

where Ψ and $\boldsymbol{\epsilon}_t$ indicate the coefficients matrices and unobservable uncertain factors, respectively. Uncertain factors follow a zero-mean normal distribution with time-invariant covariance $\mathbf{\Sigma}$. In detail, we developed the three types of demands as follows:

$$\begin{bmatrix} \zeta_t \\ \phi_t \\ \psi_t \end{bmatrix} = \begin{bmatrix} \zeta_t^0 \\ \phi_t^0 \\ \psi_t^0 \end{bmatrix} + \begin{bmatrix} \Phi_{1,1}^{t-1} & \Phi_{1,2}^{t-1} & \Phi_{1,3}^{t-1} \\ \Phi_{2,1}^{t-1} & \Phi_{2,2}^{t-1} & \Phi_{2,3}^{t-1} \\ \Phi_{3,1}^{t-1} & \Phi_{3,2}^{t-1} & \Phi_{3,3}^{t-1} \end{bmatrix} \begin{bmatrix} \zeta_{t-1} \\ \phi_{t-1} \\ \psi_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{bmatrix} \tag{4.27}$$

The demand model represented in (4.27) can be induced to the factor-based demand model presented in (4.2). If the process is stable and invertible, the VARMA process has a pure $moving - average$ (MA) representation ([64]). As See and Sim [88] and Lee and Moon [60] utilized the demand process of Graves and Stephen [41] in their computational experiments, the MA process can likewise be regarded as a factor-based demand model. We provide an example in Appendix C.2. In general, the estimation of the VARMA (p, q) process provides the correlation or covariance matrix of residuals which follows a multivariate normal distribution with a zero mean and covariance matrix $\mathbf{\Sigma}$. Solving the problem based on this normal distribution leads to a multistage stochastic optimization problem. However, estimating the expectation of the objective function in the multistage setting cannot guarantee tractability. Also, in a normal distribution, which has infinite support for both negative and positive parts, larger values above a certain level are rarely realized. Therefore, we truncated the support as $[-3\sigma, 3\sigma]$ and alleviated information about a particular distribution (See Section 4 in Ang et al. [3]). Consequently, we could obtain the inventory policy through (4.19) based on only information with a mean, support, and covariance of predefined uncertain factors.

4.5.2 Experiment 1: tractability of the RIMRSTIP

We conducted Experiment 1 to identify the tractability of the RIMRSTIP. We solved the problem by varying the planning horizon of the demand based on the randomly generated data. In other words, the data size was gradually increased to examine whether the RIMRSTIP is solvable until a modest size, or not. The results of Experiment 1 are described in Table 4.2. As presented in Table 4.2, the RIMRSTIP could solve the problem with the planning horizon from $t = 20$ to $t = 55$. Experiment 1 addresses that the robust counterpart, which is approximated from the multistage stochastic optimization model, retains the tractability. In other words, the stochastic optimization model, which is intractable, is well reformulated as the deterministic second-order cone program, which is solvable with the interior point method within the polynomial time. Analysis of the objective value lost in the approximation process was verified in Experiment 2.

Table 4.2: Results of Experiment 1 related to the tractability of the robust counterpart

	Data_Planning horizon of the data							
	A.1_20	A.2_25	A.3_30	A.4_35	A.5_40	A.6_45	A.7_50	A.8_55
Objective value	57,755	75,995	99,077	122,368	153,452	165,635	201,220	234,639
Computation time (s)	7.75	20.59	45.05	67.19	145.10	184.39	287.13	493.80

4.5.3 Experiment 2: approximation error from the expected value given perfect information

We performed Experiment 2 to verify whether the robust model provided reasonable upper bounds against the multistage stochastic optimization model, or not. Experiment 2 was conducted by increasing the length of the support of the uncertain factor from data set B.1 to B.4. We also varied the number of uncertain factors affecting the demand at time t on the same data set. To determine how the approximated objective value is affected by the uncertain factor, we utilized the concept of EV|PI. By solving the deterministic model (4.3) with the generated uncertain factors recursively, the close bound from the stochastic optimization model (4.6) can be estimated. That is, we used EV|PI as an alternative to the multistage stochastic optimization model ([3, 95, 60]). We generated the uncertain factors 10,000 times for each experiment and regarded the objective value of the stochastic optimization model (4.6) by calculating the average with the results of 10,000 times of the deterministic model (4.3). The results of Experiment 2 are presented in Table 4.3. From Table 4.3, it is evident that when the period of uncertain factors that affect the demand function became longer, the gap between the objective value from the EV|PI and LDR formulation became larger. It could also be observed that the gap increased when the length of the support of the uncertain factor increased. The re-

Table 4.3: Results of Experiment 2 relevant to the approximation error from the expected value given perfect information

		Period of uncertain factors affects the demand function				
		10	15	20	25	30
B.1	EV|PI	141,329	141,331	141,330	141,330	141,336
	LDR	143,021	143,048	143,078	143,095	143,101
	Gap (%)	1.20	1.21	1.24	1.25	1.25
B.2	EV|PI	141,354	141,355	141,354	141,352	141,351
	LDR	144,533	144,613	144,709	144,743	144,774
	Gap (%)	2.25	2.30	2.37	2.40	2.42
B.3	EV|PI	141,376	141,371	141,372	141,369	141,392
	LDR	147,104	147,367	147,858	148,221	148,349
	Gap (%)	4.05	4.24	4.59	4.85	4.92
B.4	EV|PI	141,435	141,435	141,437	141,439	141,439
	LDR	149,737	150,820	152,968	153,266	153,435
	Gap (%)	5.87	6.64	8.15	8.36	8.48

sults of Experiment 2 address that the proper adjustment of uncertain factors can reduce the loss in the approximation process.

4.5.4 Experiment 3: protection against realized uncertain factors

Since the demand models were developed as affine functions of the uncertain factors, a great variety of results can be produced in accordance with the realized uncertain factors. When the inventory manager establishes the inventory policy through the RIMRSTIP, it should protect the variability when the uncertain data is realized. To verify the robustness, we established an inventory policy based on LDR and analyzed how the objective value varies when the uncertain data is generated. As a control group for the RIMRSTIP, we made a policy that replaces order quantities based on the mean of the demand model. Recall the example of the demand model (C.2). If the inventory manager regards the demand as an expected value, the demand for the new-generation product can be estimated as $(21.46, 21.63, 21.71)$. We call the inventory policy planned in this manner as a *deterministic inventory policy* (DIP).

Based on the two types of inventory policies, LDR and DIP, we conducted simulation studies for the same data by generating uncertain factors recursively, 10,000 times. Also, we generated two types of data, C.1 and C.2, which have different support sets (the supports of C.1 are about three times larger than those of C.2). The planning horizon was set to 30, and the experiments were carried out with varying lengths of uncertain factors affecting the demand model with $t = 10$, 15, 20, 25, and 30. The results of Experiment 3 are illustrated in Figure 4.4. As shown in Figure 4.4, the objective values from the LDR formulation did not show significant variations against the realized factors. In the case of the DIP, values that deviate significantly from the average were observed for certain data. In other words, the worst-case scenario was not well protected. In practice, the inventory manager establishes the inventory policy by estimating future demand from past data. However, future data cannot be accurately predicted, which means that it is difficult to predict how uncertain data will be realized. It is also important to minimize the value of expectation, but in the event of a truly unexpected worst-case scenario, it might require the stable operation from a number of scenarios. In summary, the results of Experiment 3 examined the need for applying robust optimization to inventory management in response to correlated uncertain demands.

4.5.5 Experiment 4: differences between modeling demands from VARMA and ARMA

We conducted Experiment 4 to analyze how differences in inventory policies depend on the existence and nonexistence of inter-correlations among uncertain demands.

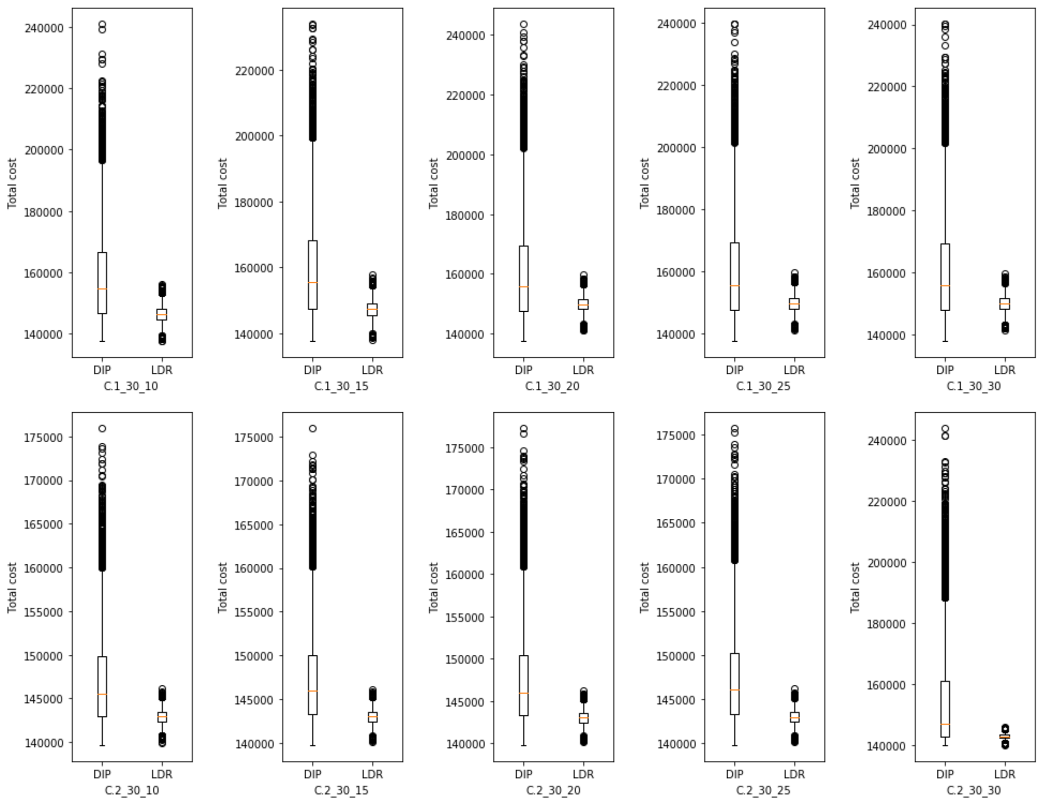


Figure 4.4: Results of Experiment 3, protection against realized random factors

We compared the demand model from the VARMA model, which captures the interrelationship among three demands, with the ARMA model, which is affected only by its historical data. We refer to the inventory policies obtained through the VARMA and ARMA process as the $CORR$ and $IDPT$ policies, respectively (We used CORR and IDPT to emphasize the correlation across the demands and independent among the demands). We performed simulation experiments with these two types of inventory policies, which were obtained through the LDR formulation. In a similar manner to Experiment 3, simulation studies were conducted based on the 10,000 sets of randomly generated values for the uncertain factors. The results of Exper-

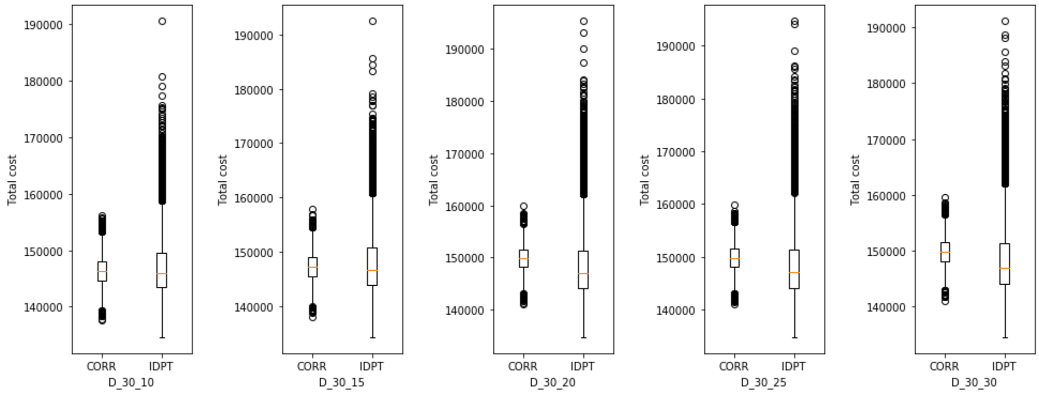


Figure 4.5: Results of Experiment 4, differences between modeling demands from VARMA and ARMA

iment 4 are shown in Figure 4.5. As can be observed in Figure 4.5, the inventory policy from the CORR provides stable and robust solutions. The difference between the minimum and maximum of the objective value is significantly smaller than that of IDPT. However, the average value of the IDPT was smaller than that of CORR. It was also observed that the lowest cost occurred in IDPT.

4.5.6 Experiments 5 and 6: comparisons of backlogged refurbishment service with or without trade-in program

We performed Experiment 5 to identify the effect of the trade-in program on the acquisition of the returned products required for the refurbishment service. As an experimental group, we introduced TIP, which is a supply chain system to which the trade-in program is considered. As a control group, we introduced NO_TIP where the trade-in program is not considered (See Figure C.1 in Appendix C.1). Four types of data sets, E.1 to E.4, were generated randomly based on periods 20, 25, 30, and 35, respectively. We first derived the order policy based on the LDR and conducted the simulation experiments by inputting the realized uncertain factors 10,000 times. Unlike other experiments in this study, which calculated the total cost over the entire planning horizon, Experiment 5 computed the sum of backlogged inventories over the entire period. The mean, minimum, and maximum values of the backlogged inventories for the 10,000 times simulation experiments are listed in Table 4.4 as Avg, Min, and Max, respectively. From Table 4.4, it can be identified that the introduction of the trade-in program reduced the backlog of the refurbishment service. The backlog of the refurbishment service not only incurs a high penalty cost but also triggers the production of old-generation products, which may cause inefficiency in producing new-generation products. Experiment 5 addresses that the introduction of a trade-in program increases the complexity of the decision-making process for retailers; however, efficient management can lead to a stable operation that better meets customer needs in terms of refurbishment services.

Experiment 6 was then conducted to figure out how the trade-in program affects the input of the products used for the refurbishment service. Recall that the production of the old-generation products, refurbishing process, and remanufacturing process are input in the balance equation relevant to the refurbishment service (Figure 4.3). In the same manner as Experiment 5, 10,000 times simulation experiments were conducted to identify the variations with respect to the production amounts of the refurbishment service. The total production amount of the refurbishing, remanufacturing, and old-generation products, which are defined as RF, RM, and OG, respectively, are provided in Table 4.5. As indicated in Table 4.5, the production of the old-generation product was not featured in the TIP system. Meanwhile, the remanufacturing process was not featured in NO_TIP. The introduction of the

Table 4.4: Results of Experiment 5, the effect of the trade-in program on the acquisition of the returned products

	E.1			E.2			E.3			E.4		
	Avg	Min	Max	Avg	Min	Max	Avg	Min	Max	Avg	Min	Max
TIP	79	67	92	90	78	103	72	61	83	57	43	71
NO_TIP	312	280	343	356	321	396	343	314	374	276	241	313

Table 4.5: Results of Experiment 6, the effect of the trade-in program on the input of the products used for the refurbishment service

		E.1			E.2			E.3			E.4		
		Avg	Min	Max	Avg	Min	Max	Avg	Min	Max	Avg	Min	Max
TIP	OG	-	-	-	-	-	-	-	-	-	-	-	-
	RM	535	483	582	420	379	466	518	465	564	685	631	743
	RF	562	529	592	896	832	950	1134	1081	1195	1291	1236	1358
NO_TIP	OG	232	223	242	290	276	304	306	294	317	335	342	369
	RM	849	776	911	1027	958	1106	1345	1258	1444	1615	1537	1715
	RF	-	-	-	-	-	-	-	-	-	-	-	-

trade-in program reduced the production of the remanufacturing processes and old-generation products, and increased the production of the refurbishing processes. Since the costs and lead times for each process are assumed as $RF < RM < OG$, the introduction of the trade-in program has enabled efficient and stable operation in terms of inventory management.

4.6 Summary

In this study, we considered the multiperiod inventory model which incorporates the refurbishment service and trade-in program simultaneously. Due to the reverse flows of the products, the inventory model features a closed-loop supply chain system. To capture the correlations among the three types of demands, including new-generation, refurbishment service, and trade-in program demands, we adopted the factor-based demand model, which only requires the first and second moments of the uncertain factors. By approximating the upper bound of the multistage stochastic optimization model, we could derive the robust counterpart. The computational results showed that the robust counterpart retains tractability until a modest size. It also provided the reasonable upper bound from the multistage stochastic optimization model. Additionally, this study offered managerial insights, which could be instructive to inventory managers. By analyzing the results of the computational experiments, we derived the following managerial insights as follows:

(i) The inventory policy derived from the LDR formulation provides robust and stable solutions. Meanwhile, the accuracy lost through approximation was reasonable and tractability was greatly improved. That is, the robust model provided a much better policy than the deterministic model and was much more efficient than the multistage stochastic optimization model.

(ii) Depending on the propensity of the inventory manager, different inventory policies might be preferred. If the inventory manager pursues to minimize the expected cost for various scenarios, it would be better to establish an inventory policy through the IDPT, which incorporates only its own past data.

(iii) If the inventory manager seeks to operate in a more stable manner, it would be better to establish an inventory policy through the CORR, which incorporates the inter-correlation among demands as well as from its own past data.

(iv) The introduction of a trade-in program not only plays a role in a sales promotion to increase customer demand but also plays a role in the efficient acquisition of the returned products for a refurbishment service. By reducing the number of backlogged inventories for the refurbishment service, the retailer can improve a customer's service level. Furthermore, reducing the number of backlogged inventories mitigates the inefficient production of discontinued old-generation products and enhances the refurbishment process of products that can be produced efficiently.

Chapter 5

Conclusions

5.1 Summary

Nowadays, sales promotion becomes crucial to the successful operation of the company. It is well known that the proper introduction of sales promotion brings significant benefits to the company. However, the introduction of sales promotion increases the uncertainty and complexity of decision making from a perspective of the company's inventory manager. Therefore, a more advanced method of inventory management is required.

In this dissertation, we considered three inventory models with sales promotions: (i) newsvendor model considering the start time of markdown sale, (ii) multiperiod inventory model with BOGO, and (iii) multiperiod inventory model with a trade-in program. For the markdown sale, we introduced the concept of the start time of markdown sale in the newsvendor model. We developed the bi-section method algorithm to obtain the optimal combination of the start time of the markdown sale and order quantity. We also proposed the revenue-sharing contract to reach the global optimum from the system perspective. The supply chain is coordinated with the revenue-sharing contract, which leads to the coincidence of both the start time of markdown sale and order quantity. As shown in the numerical example, the revenue-sharing contract increased the profits of the retailer and manufacturer.

For the inventory model with BOGO, we adopted the concept of the mobile application MOR. We modeled the uncertain revisiting rate of the customer as affinely dependent on the uncertain factors. We utilized the affinely adjustable robust optimization approach based on the linear decision rule to deal with the uncertain factors in the multistage stochastic optimization model. By approximating the upper bound of the multistage stochastic optimization model, we derived the robust counterpart that is tractable and provides the reasonable upper bound.

In the case of the inventory model with the trade-in program, a closed-loop supply chain system is considered. This inventory model incorporates the uncertain demands of the new-generation product, trade-in program, and refurbishment service, simultaneously. To capture the correlations of these uncertain demands, we adopted the factor-based demand model. To overcome the intractability of the multistage stochastic optimization model, we adopted the linear decision rule. The upper bound of the expectation of positive parts in the objective function is derived to the second-order cone program. Accordingly, we derived the robust counterpart by approximating the multistage stochastic optimization model. Computational results showed that the proposed robust model is tractable and provides a robust solution.

5.2 Future research

In this dissertation, three types of inventory models, considering three different sales promotions, are studied. Since the research related to the inventory management when the sales promotions are implemented, was not studied sufficiently, there is enormous potential for future research.

We obtained several insights by adding the notion of a start time of the mark-down sale in the newsvendor model. Since the research relevant to this concept is conducted insufficiently, plenty of variations can be considered. In terms of demand modeling, several methodologies could lead to practical results. Because this study is extended based on the single-period newsvendor model, new insights can be found by extending the model to the multi-period, multi-item, multi-retailer, or multi-echelon model. That is, this study is expected to be a cornerstone of numerous upcoming future studies.

In the case of the BOGO model, to the best of our knowledge, this research is the first attempt to develop an IMMOR by adopting the distributionally robust optimization approach. Therefore, the opportunity for future research is immense. Naturally, an extension to a multi-item inventory model could be considered. All items could be MOR-based BOGO products or a mix of products that are not part of the promotion. RIMMOR could also be generalized as a buy-x-get-y promotion, or offering other freebies, rather than the BOGO promotion ([48]). When the model is generalized, it will give flexibility to the retailer's decision. The promotion, which returns the point to the customer, which has a similar mechanism with BOGO, can be considered. Moon et al. [69] explored this issue with supply chain coordination. If there is an expiry date of available point, and the customer uses the returned point to purchase an additional product, the inventory model presented in this study could be extended. A study on the dynamic pricing of BOGO products was conducted by Kim et al. [55]. If MOR is applied, a model simultaneously considering both pricing and inventory can be developed.

In practice, a retailer operates the trade-in program with an online store as well as with a physical "bricks and mortar" store. For future research, an inventory model considering the trade-in program can be extended to a multi-channel or an omnichannel system. Alternatively, the trade-in program can be run by returning points to the customer with the same function as cash. In this case, a wide range of purchasing options would be available for the customers rather than the existing and more limiting option of purchasing discounted new-generation products. Considering the uncertain environment, we focused on the uncertainty in demands. Depending on the condition of returned products, the cost of refurbishing or remanufacturing process can vary. In this study, we divided the condition of the returned products into two types and considered them as deterministic manners. For further study, the condition of returned products could be regarded as uncertain values.

Bibliography

[1] V. V. Agrawal, M. Ferguson, and G. C. Souza, *Trade-in rebates for price discrimination and product recovery*, IEEE Transactions on Engineering Management, 63 (2016), pp. 326–339.

[2] F. Alizadeh and D. Goldfarb, *Second-order cone programming*, Mathematical programming, 95 (2003), pp. 3–51.

[3] M. Ang, Y. F. Lim, and M. Sim, *Robust storage assignment in unit-load warehouses*, Management Science, 58 (2012), pp. 2114–2130.

[4] T. Avinadav and T. Arponen, *An eoq model for items with a fixed shelf-life and a declining demand rate based on time-to-expiry technical note*, Asia-Pacific Journal of Operational Research, 26 (2009), pp. 759–767.

[5] Y. Aviv and A. Pazgal, *Optimal pricing of seasonal products in the presence of forward-looking consumers*, Manufacturing & Service Operations Management, 10 (2008), pp. 339–359.

[6] H. Bae, I. Moon, and W. Yun, *A time-varying lot sizes approach for the economic lot scheduling problem with returns*, International Journal of Production Research, 54 (2016), pp. 3380–3396.

[7] A. Ben-Tal, G. Boaz, and S. Shimrit, *Robust multi-echelon multi-period inventory control*, European Journal of Operational Research, 199 (2009), pp. 922–935.

[8] A. Ben-Tal, B. Golany, A. Nemirovski, and J.-P. Vial, *Retailer-supplier flexible commitments contracts: A robust optimization approach*, Manufacturing & Service Operations Management, 7 (2005), pp. 248–271.

[9] A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski, *Adjustable robust solutions of uncertain linear programs*, Mathematical Programming, 99 (2004), pp. 351–376.

[10] A. Ben-Tal and A. Nemirovski, *Robust convex optimization*, Mathematics of Operations Research, 23 (1998), pp. 769–805.

[11] ——, *Robust solutions of uncertain linear programs*, Operations research letters, 25 (1999), pp. 1–13.

[12] ——, *Robust solutions of linear programming problems contaminated with uncertain data*, Mathematical programming, 88 (2000), pp. 411–424.

[13] D. Bertsimas, D. B. Brown, and C. Caramanis, *Theory and applications of robust optimization*, SIAM review, 53 (2011), pp. 464–501.

[14] D. Bertsimas and M. Sim, *The price of robustness*, Operations Research, 52 (2004), pp. 35–53.

[15] D. Bertsimas and A. Thiele, *A robust optimization approach to inventory theory*, Operations Research, 54 (2006), pp. 150–168.

[16] Y. Bian, J. Xie, T. W. Archibald, and Y. Sun, *Optimal extended warranty strategy: Offering trade-in service or not?*, European Journal of Operational Research, 278 (2019), pp. 240–254.

[17] I. Biswas and B. Avittathur, *The price-setting limited clearance sale inventory model*, Annals of Operations Research, (2018), pp. 1–17.

[18] G. H. Brundtland, M. Khalid, S. Agnelli, S. Al-Athel, and B. Chidzero, *Our common future*, New York: Oxford University Press, (1987).

[19] G. P. Cachon and M. A. Lariviere, *Supply chain coordination with revenue-sharing contracts: strengths and limitations*, Management Science, 51 (2005), pp. 30–44.

[20] K. Cao, J. Wang, G. Dou, and Q. Zhang, *Optimal trade-in strategy of retailers with online and offline sales channels*, Computers & Industrial Engineering, 123 (2018), pp. 148–156.

[21] K. Cao, X. Xu, Y. Bian, and Y. Sun, *Optimal trade-in strategy of business-to-consumer platform with dual-format retailing model*, Omega, 82 (2019), pp. 181–192.

[22] L. E. Cárdenas-Barrón, K.-J. Chung, and G. Treviño-Garza, *Celebrating a century of the economic order quantity model in honor of ford whitman harris*, International Journal of Production Economics, 155 (2014), pp. 1–7.

[23] L. E. Cárdenas-Barrón, J. L. González-Velarde, and G. Treviño-Garza, *A new approach to solve the multi-product multi-period inventory lot sizing with supplier selection problem*, Computers & Operations Research, 64 (2015), pp. 225–232.

[24] L. E. Cárdenas-Barrón, J. L. González-Velarde, G. Treviño-Garza, and D. Garza-Nuñez, *Heuristic algorithm based on reduce and optimize approach for a selective and periodic inventory routing problem in a waste vegetable oil collection environment*, International Journal of Production Economics, 211 (2019), pp. 44–59.

[25] J.-M. Chen and Y.-T. Hsu, *Trade-in strategy for a durable goods firm with recovery cost*, Journal of Industrial and Production Engineering, 32 (2015), pp. 396–407.

[26] W. Chen and M. Sim, *Goal-driven optimization*, Operations Research, 57 (2009), pp. 342–357.

[27] X. Chen, S. X. Zhou, and Y. Chen, *Integration of inventory and pricing decisions with costly price adjustments*, Operations Research, 59 (2011), pp. 1144–1158.

[28] W. Chung, S. Talluri, and R. Narasimhan, *Optimal pricing and inventory strategies with multiple price markdowns over time*, European Journal of Operational Research, 243 (2015), pp. 130–141.

[29] P. De Giovanni and G. Zaccour, *Optimal quality improvements and pricing strategies with active and passive product returns*, Omega, 88 (2019), pp. 248–262.

[30] R. Declaration, *Rio declaration on environment and development*, In Report of the United Nations Conference on Environment and Development, Rio De Janeiro, (1992), pp. 3–14.

[31] M. Denizel, M. Ferguson, et al., *Multiperiod remanufacturing planning with uncertain quality of inputs*, IEEE Transactions on Engineering Management, 57 (2009), pp. 394–404.

[32] W. Elmaghraby and P. Keskinocak, *Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions*, Management Science, 49 (2003), pp. 1287–1309.

[33] R. Eslamipoor, M. Fakhrzad, and Y. Zare Mehrjerdi, *A new robust optimization model under uncertainty for new and remanufactured products*, International Journal of Management Science and Engineering Management, 10 (2015), pp. 137–143.

[34] A. Federgruen and A. Heching, *Combined pricing and inventory control under uncertainty*, Operations Research, 47 (1999), pp. 454–475.

[35] V. Gabrel, C. Murat, and A. Thiele, *Recent advances in robust optimization: An overview*, European Journal of Operational Research, 235 (2014), pp. 471–483.

[36] J. Gaur, M. Amini, and A. Rao, *Closed-loop supply chain configuration for new and reconditioned products: An integrated optimization model*, Omega, 66 (2017), pp. 212–223.

[37] J. Goh and M. Sim, *Robust optimization made easy with rome*, Operations Research, 59 (2011), pp. 973–985.

[38] B. L. Gorissen, İ. Yanikoğlu, and D. den Hertog, *A practical guide to robust optimization*, Omega, 53 (2015), pp. 124–137.

[39] K. Govindan, H. Soleimani, and D. Kannan, *Reverse logistics and closed-loop supply chain: A comprehensive review to explore the future*, European journal of operational research, 240 (2015), pp. 603–626.

[40] D. Granot and S. Yin, *Price and order postponement in a decentralized newsvendor model with multiplicative and price-dependent demand*, Operations Research, 56 (2008), pp. 121–139.

[41] S. C. Graves, *A single-item inventory model for a nonstationary demand process*, Manufacturing & Service Operations Management, 1 (1999), pp. 50–61.

[42] D. Gupta, A. V. Hill, and T. Bouzdine-Chameeva, *A pricing model for clearing end-of-season retail inventory*, European Journal of Operational Research, 170 (2006), pp. 518–540.

[43] X. Han, Q. Yang, J. Shang, and X. Pu, *Optimal strategies for trade-old-for-remanufactured programs: Receptivity, durability, and subsidy*, International Journal of Production Economics, 193 (2017), pp. 602–616.

[44] A. Hasani, S. H. Zegordi, and E. Nikbakhsh, *Robust closed-loop supply chain network design for perishable goods in agile manufacturing under uncertainty*, International Journal of Production Research, 50 (2012), pp. 4649–4669.

[45] A. Hassanpour, J. Bagherinejad, and M. Bashiri, *A robust leader-follower approach for closed loop supply chain network design considering returns quality levels*, Computers & Industrial Engineering, 136 (2019), pp. 293–304.

[46] X. Hong, L. Wang, Y. Gong, and W. Chen, *What is the role of value-added service in a remanufacturing closed-loop supply chain?*, International Journal of Production Research, (2019), pp. 1–20.

[47] P. Hu, S. Shum, and M. Yu, *Joint inventory and markdown management for perishable goods with strategic consumer behavior*, Operations Research, 64 (2015), pp. 118–134.

[48] K.-L. Huang, C.-W. Kuo, and H.-J. Shih, *Advance selling with freebies and limited production capacity*, Omega, 73 (2017), pp. 18–28.

[49] Y. Huang, *A closed-loop supply chain with trade-in strategy under retail competition*, Mathematical Problems in Engineering, 2018 (2018).

[50] D. Ivanov, A. Tsipoulanidis, J. Schönberger, et al., *Global supply chain and operations management*, A Decision-Oriented Introduction to the Creation of Value, (2017).

[51] A. Jabbarzadeh, M. Haughton, and A. Khosrojerdi, *Closed-loop supply chain network design under disruption risks: A robust approach with real world application*, Computers & Industrial Engineering, 116 (2018), pp. 178–191.

[52] B. Kazaz and S. Webster, *Price-setting newsvendor problems with uncertain supply and risk aversion*, Operations Research, 63 (2015), pp. 807–811.

[53] M. Khouja, *The single-period (news-vendor) problem: literature review and suggestions for future research*, Omega, 27 (1999), pp. 537–553.

[54] J. Kim, B. Do Chung, Y. Kang, and B. Jeong, *Robust optimization model for closed-loop supply chain planning under reverse logistics flow and demand uncertainty*, Journal of Cleaner Production, 196 (2018), pp. 1314–1328.

[55] K.-K. Kim, C.-G. Lee, and S. Park, *Dynamic pricing with 'bogo'promotion in revenue management*, International Journal of Production Research, 54 (2016), pp. 5283–5302.

[56] A. Kocabiyikoğlu and I. Popescu, *An elasticity approach to the newsvendor with price-sensitive demand*, Operations research, 59 (2011), pp. 301–312.

[57] O. Kwon, A. J. Dukes, S. Siddarth, and J. M. Silva-Risso, *The informational role of product trade-ins for pricing durable goods*, The Journal of Industrial Economics, 63 (2015), pp. 736–762.

[58] G. J. Kyparisis and C. Koulamas, *The price-setting newsvendor problem with nonnegative linear additive demand*, European Journal of Operational Research, 269 (2018), pp. 695–698.

[59] A. H.-L. Lau and H.-S. Lau, *The newsboy problem with price-dependent demand distribution*, IIE transactions, 20 (1988), pp. 168–175.

[60] S. Lee and I. Moon, *Robust empty container repositioning considering foldable containers*, European Journal of Operational Research, 280 (2020), pp. 909–925.

[61] K. J. Li, D. K. Fong, and S. H. Xu, *Managing trade-in programs based on product characteristics and customer heterogeneity in business-to-business markets*, Manufacturing & Service Operations Management, 13 (2011), pp. 108–123.

[62] K. J. Li and S. H. Xu, *The comparison between trade-in and leasing of a product with technology innovations*, Omega, 54 (2015), pp. 134–146.

[63] X. Li, F. Baki, P. Tian, and B. A. Chaouch, *A robust block-chain based tabu search algorithm for the dynamic lot sizing problem with product returns and remanufacturing*, Omega, 42 (2014), pp. 75–87.

[64] H. Lütkepohl, *New introduction to multiple time series analysis*, Springer Science & Business Media, 2005.

[65] P. Ma, Y. Gong, and P. Mirchandani, *Trade-in for remanufactured products: Pricing with double reference effects*, Working paper, (2019).

[66] E. Mardan, K. Govindan, H. Mina, and S. M. Gholami-Zanjani, *An accelerated benders decomposition algorithm for a bi-objective green closed loop supply chain network design problem*, Journal of Cleaner Production, (2019).

[67] S. Mitra, *Newsvendor problem with clearance pricing*, European Journal of Operational Research, 268 (2018), pp. 193–202.

[68] S. Mitra and A. K. Chatterjee, *Single-period newsvendor problem under random end-of-season demand*, Journal of the Operational Research Society, 69 (2018), pp. 1046–1060.

[69] I. Moon, J. Xu, X. Feng, and X. Ruan, *Cooperative sales promotion with a point-sharing policy: Advantages and limitations*, Omega, (2019, to be appeared).

[70] I. Moon, D. K. Yoo, and S. Saha, *The distribution-free newsboy problem with multiple discounts and upgrades*, Mathematical Problems in Engineering, 2016 (2016).

[71] D. Nakandala, H. Lau, and P. K. Shum, *A lateral transshipment model for perishable inventory management*, International Journal of Production Research, 55 (2017), pp. 5341–5354.

[72] I. Olkin, *Range restrictions for product-moment correlation matrices*, Psychometrika, 46 (1981), pp. 469–472.

[73] E. Özceylan and T. Paksoy, *A mixed integer programming model for a closed-loop supply-chain network*, International Journal of Production Research, 51 (2013), pp. 718–734.

[74] S. Panda, N. M. Modak, S. S. Sana, and M. Basu, *Pricing and replenishment policies in dual-channel supply chain under continuous unit cost decrease*, Applied Mathematics and Computation, 256 (2015), pp. 913–929.

[75] M. Pesendorfer, *Retail sales: A study of pricing behavior in supermarkets*, The Journal of Business, 75 (2002), pp. 33–66.

[76] N. C. Petruzzi and M. Dada, *Pricing and the newsvendor problem: A review with extensions*, Operations Research, 47 (1999), pp. 183–194.

[77] M. S. Pishvaee, M. Rabbani, and S. A. Torabi, *A robust optimization approach to closed-loop supply chain network design under uncertainty*, Applied Mathematical Modelling, 35 (2011), pp. 637–649.

[78] L. H. Polatoglu, *Optimal order quantity and pricing decisions in single-period inventory systems*, International Journal of Production Economics, 23 (1991), pp. 175–185.

[79] Y. Qin, R. Wang, A. J. Vakharia, Y. Chen, and M. M. Seref, *The newsvendor problem: Review and directions for future research*, European Journal of Operational Research, 213 (2011), pp. 361–374.

[80] H. Rahimian and S. Mehrotra, *Distributionally robust optimization: A review*, arXiv preprint arXiv:1908.05659, (2019).

[81] R. S. Rao, O. Narasimhan, and G. John, *Understanding the role of trade-ins in durable goods markets: Theory and evidence*, Marketing Science, 28 (2009), pp. 950–967.

[82] S. Ray, T. Boyaci, and N. Aras, *Optimal prices and trade-in rebates for durable, remanufacturable products*, Manufacturing & Service Operations Management, 7 (2005), pp. 208–228.

[83] J. Rubio-Herrero and M. Baykal-Gürsoy, *On the unimodality of the price-setting newsvendor problem with additive demand under risk considerations*, European Journal of Operational Research, 265 (2018), pp. 962–974.

[84] S. Russ, *A translation of bolzano's paper on the intermediate value theorem*, Historia Mathematica, 7 (1980), pp. 156–185.

[85] B. Sarkar, *Mathematical and analytical approach for the management of defective items in a multi-stage production system*, Journal of cleaner production, 218 (2019), pp. 896–919.

[86] B. Sarkar, R. Guchhait, M. Sarkar, and L. E. Cárdenas-Barrón, *How does an industry manage the optimum cash flow within a smart production system with the carbon footprint and carbon emission under logistics framework?*, International Journal of Production Economics, 213 (2019), pp. 243–257.

[87] B. Schulte and A.-L. Sachs, *The price-setting newsvendor with poisson demand*, European Journal of Operational Research, 283 (2020), pp. 125–137.

[88] C.-T. See and M. Sim, *Robust approximation to multiperiod inventory management*, Operations Research, 58 (2010), pp. 583–594.

[89] K. Shampanier, N. Mazar, and D. Ariely, *Zero as a special price: The true value of free products*, Marketing Science, 26 (2007), pp. 742–757.

[90] A. Shapiro, *Inference of statistical bounds for multistage stochastic programming problems*, Mathematical Methods of Operations Research, 58 (2003), pp. 57–68.

[91] ——, *Stochastic programming approach to optimization under uncertainty*, Mathematical Programming, 112 (2008), pp. 183–220.

[92] A. Shapiro and A. Nemirovski, *On complexity of stochastic programming problems*, in Continuous optimization, Springer, 2005, pp. 111–146.

[93] A. Shapiro and A. Philpott, *A tutorial on stochastic programming*, Manuscript. Available at www2. isye. gatech. edu/ashapiro/publications. html, 17 (2007).

[94] J.-B. Sheu and T.-M. Choi, *Extended consumer responsibility: Syncretic value-oriented pricing strategies for trade-in-for-upgrade programs*, Transportation Research Part E: Logistics and Transportation Review, 122 (2019), pp. 350–367.

[95] Y. Shin, S. Lee, and I. Moon, *Robust multiperiod inventory model with a new type of buy one get one promotion: "my own refrigerator"*, Omega, (2019), p. 102170.

[96] S. A. Smith and D. D. Achabal, *Clearance pricing and inventory policies for retail chains*, Management Science, 44 (1998), pp. 285–300.

[97] G. C. Souza, *Closed-loop supply chains with remanufacturing*, in State-of-the-Art Decision-Making Tools in the Information-Intensive Age, Informs, Hanover, MD, 2008, pp. 130–153.

[98] ——, *Closed-loop supply chains with remanufacturing*, in State-of-the-Art Decision-Making Tools in the Information-Intensive Age, Informs, 2008, pp. 130–153.

[99] ——, *Closed-loop supply chains: a critical review, and future research*, Decision Sciences, 44 (2013), pp. 7–38.

[100] A. L. Soyster, *Convex programming with set-inclusive constraints and applications to inexact linear programming*, Operations Research, 21 (1973), pp. 1154–1157.

[101] M. Talaei, B. F. Moghaddam, M. S. Pishvaee, A. Bozorgi-Amiri, and S. Gholamnejad, *A robust fuzzy optimization model for carbon-efficient closed-loop supply chain network design problem: a numerical illustration in electronics industry*, Journal of Cleaner Production, 113 (2016), pp. 662–673.

[102] A. A. Taleizadeh, M. S. Moshtagh, and I. Moon, *Optimal decisions of price, quality, effort level and return policy in a three-level closed-loop supply chain based on different game theory approaches*, European Journal of Industrial Engineering, 11 (2017), pp. 486–525.

[103] O. Tang and R. Teunter, *Economic lot scheduling problem with returns*, Production and Operations Management, 15 (2006), pp. 488–497.

[104] Y. Tang and C. Li, *Uncertainty management in remanufacturing: a review*, in 2012 IEEE International Conference on Automation Science and Engineering (CASE), IEEE, 2012, pp. 52–57.

[105] R. H. Teunter, Z. P. Bayindir, and W. V. Den Heuvel, *Dynamic lot sizing with product returns and remanufacturing*, International Journal of Production Research, 44 (2006), pp. 4377–4400.

[106] H.-T. Tsang and H.-Y. Mak, *Robust optimization approach to empty container repositioning in liner shipping*, in Handbook of Ocean Container Transport Logistics, Springer, 2015, pp. 209–229.

[107] M. Ullah and B. Sarkar, *Recovery-channel selection in a hybrid manufacturing-remanufacturing production model with rfid and product quality*, International Journal of Production Economics, 219 (2020), pp. 360–374.

[108] A. Van Ackere and D. J. Reyniers, *A rationale for trade-ins*, Journal of Economics and Business, 45 (1993), pp. 1–16.

[109] ——, *Trade-ins and introductory offers in a monopoly*, The RAND Journal of Economics, (1995), pp. 58–74.

[110] ——, *Trade-ins and introductory offers in a monopoly*, The RAND Journal of Economics, (1995), pp. 58–74.

[111] C. Wei, Y. Li, and X. Cai, *Robust optimal policies of production and inventory with uncertain returns and demand*, International Journal of Production Economics, 134 (2011), pp. 357–367.

[112] T. M. Whitin, *Inventory control and price theory*, Management science, 2 (1955), pp. 61–68.

[113] M. Xu, Y. F. Chen, and X. Xu, *The effect of demand uncertainty in a price-setting newsvendor model*, European Journal of Operational Research, 207 (2010), pp. 946–957.

[114] M. Xu and Y. Lu, *The effect of supply uncertainty in price-setting newsvendor models*, European Journal of Operational Research, 227 (2013), pp. 423–433.

[115] N. Yang and R. Zhang, *Dynamic pricing and inventory management under inventory-dependent demand*, Operations Research, 62 (2014), pp. 1077–1094.

[116] P.-C. Yang, H.-M. Wee, S.-L. Chung, and Y.-Y. Huang, *Pricing and replenishment strategy for a multi-market deteriorating product with time-varying and price-sensitive demand*, Journal of Industrial & Management Optimization, 9 (2013), pp. 769–787.

[117] İ. Yanikoğlu, B. L. Gorissen, and D. den Hertog, *A survey of adjustable robust optimization*, European Journal of Operational Research, 277 (2019), pp. 799–813.

[118] L. Yao, Y. F. Chen, and H. Yan, *The newsvendor problem with pricing: Extensions*, International Journal of Management Science and Engineering Management, 1 (2006), pp. 3–16.

[119] M. Yavari and M. Geraeli, *Heuristic method for robust optimization model for green closed-loop supply chain network design of perishable goods*, Journal of Cleaner Production, 226 (2019), pp. 282–305.

[120] T. Ye and H. Sun, *Price-setting newsvendor with strategic consumers*, Omega, 63 (2016), pp. 103–110.

[121] R. Yin, H. Li, and C. S. Tang, *Optimal pricing of two successive-generation products with trade-in options under uncertainty*, Decision Sciences, 46 (2015), pp. 565–595.

[122] R. Yin and C. S. Tang, *Optimal temporal customer purchasing decisions under trade-in programs with up-front fees*, Decision Sciences, 45 (2014), pp. 373–400.

[123] P. You, *Inventory policy for products with price and time-dependent demands*, Journal of the Operational Research Society, 56 (2005), pp. 870–873.

[124] L. Young, *Price, inventory and the structure of uncertain demand*, New Zealand Operations Research, 6 (1978), pp. 157–177.

[125] E. Zabel, *Monopoly and uncertainty*, The Review of Economic Studies, 37 (1970), pp. 205–219.

[126] S. Zanoni, A. Segerstedt, O. Tang, and L. Mazzoldi, *Multi-product economic lot scheduling problem with manufacturing and remanufacturing using a basic period policy*, Computers & Industrial Engineering, 62 (2012), pp. 1025–1033.

[127] F. Zhang and R. Zhang, *Trade-in remanufacturing, customer purchasing behavior, and government policy*, Manufacturing & Service Operations Management, 20 (2018), pp. 601–616.

[128] X. Zhang, G. Zhao, Y. Qi, and B. Li, *A robust fuzzy optimization model for closed-loop supply chain networks considering sustainability*, Sustainability, 11 (2019), p. 5726.

[129] Y.-C. Zhou and X.-C. Sun, *Robust optimal inventory and acquisition effort decisions in a hybrid manufacturing/remanufacturing system*, Journal of Industrial and Production Engineering, 36 (2019), pp. 335–350.

[130] R. Zhu, X. Chen, and S. Dasgupta, *Can trade-ins hurt you? exploring the effect of a trade-in on consumers' willingness to pay for a new product*, Journal of Marketing Research, 45 (2008), pp. 159–170.

[131] X. Zhu, M. Wang, G. Chen, and X. Chen, *The effect of implementing trade-in strategy on duopoly competition*, European Journal of Operational Research, 248 (2016), pp. 856–868.

Appendix A

A.1

$$
\begin{aligned}
&\mathbb{E}[\min(q,D)] = \mathbb{E}[D|D \le q] \cdot Pr(D \le q) + \mathbb{E}[q|D \ge q] \cdot Pr(D \ge q) \\
&= \mathbb{E}[\frac{t_m}{T}(a-bp) + \frac{t_m}{T}\epsilon | \frac{t_m}{T}(a-bp) + \frac{t_m}{T}\epsilon \le q] \cdot Pr(\frac{t_m}{T}(a-bp) + \frac{t_m}{T}\epsilon \le q) \\
&+ \mathbb{E}[q| \frac{t_m}{T}(a-bp) + \frac{t_m}{T}\epsilon \ge q] \cdot Pr(\frac{t_m}{T}(a-bp) + \frac{t_m}{T}\epsilon \ge q) \\
&= \frac{t_m}{T} \cdot (a-bp) \cdot F(\frac{T}{t_m}q - (a-bp)) + \frac{t_m}{T} \cdot \mathbb{E}[\epsilon|\epsilon \le \frac{T}{t_m}q - (a-bp)] \cdot Pr(\epsilon \le \frac{T}{t_m}q - (a-bp)) \\
&+ q \cdot Pr(\epsilon \ge \frac{T}{t_m}q - (a-bp)) = \frac{t_m}{T}(a-bp) \cdot F(\frac{T}{t_m}q - (a-bp)) + \frac{t_m}{T} \cdot \frac{\int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx}{Pr(\epsilon \le \frac{T}{t_m}q - (a-bp))} \\
&+ q \cdot [1 - F(\frac{T}{t_m}q - (a-bp))] = \frac{t_m}{T} \cdot (a-bp)F(\frac{T}{t_m}q - (a-bp)) + \frac{t_m}{T}\int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx \\
&+ q \cdot [1 - F(\frac{T}{t_m}q - (a-bp))]
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\min[\max(q-D,0),D']\right] = \\
&Pr(D \le q \le D+D') \cdot (q - \mathbb{E}[D|D \le q \le D+D']) + Pr(D+D' \le q) \cdot \mathbb{E}[D' \mid D+D' \le q] \\
&= Pr\left(\frac{t_m}{T}(a-bp) + \frac{t_m}{T}\epsilon \le q \le a - bp + \epsilon + \frac{T-t_m}{T}\alpha bp\right) \cdot \\
&\left(q - \mathbb{E}\left[\frac{t_m}{T}(a-(1-\alpha)\cdot bp) + \frac{T-t_m}{T}\epsilon \mid a - bp + \epsilon + \frac{T-t_m}{T}\alpha bp\right]\right) \\
&+ Pr(a - bp + \epsilon + \frac{T-t_m}{T}\alpha bp \le q) \cdot \\
&\mathbb{E}\left[\frac{T-t_m}{T}(a-(1-\alpha)\cdot bp) + \frac{T-t_m}{T}\epsilon \mid a - bp + \epsilon + \frac{T-t_m}{T}\alpha bp \le q\right]
\end{aligned}
$$

where

$$
\begin{aligned}
&Pr\left(\frac{t_m}{T}(a-bp)+\frac{t_m}{T}\epsilon \leq q \leq a-bp+\epsilon+\frac{T-t_m}{T}\alpha bp\right)\cdot \\
&\left(q-\mathbb{E}\left[\frac{t_m}{T}(a-bp)+\frac{t_m}{T}\epsilon \mid \frac{t_m}{T}(a-bp)+\frac{t_m}{T}\epsilon \leq q \leq a-bp+\epsilon+\frac{T-t_m}{T}\alpha bp\right]\right) \\
&= Pr\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp \leq \epsilon \leq \frac{T}{t_m}q-(a-bp)\right)\cdot\left(q-\frac{t_m}{T}(a-bp)\right) \\
&-\frac{t_m}{T}\int_{q-(a-bp)-\frac{T-t_m}{T}\alpha bp}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx \\
&= \left(q-\frac{t_m}{T}(a-bp)\right)\cdot\left(F\left(\frac{T}{t_m}q-(a-bp)\right)-F\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)\right) \\
&-\frac{t_m}{T}\int_{q-(a-bp)-\frac{T-t_m}{T}\alpha bp}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx
\end{aligned}
$$

and

$$
\begin{aligned}
&Pr\left(a-bp+\epsilon+\frac{T-t_m}{T}\alpha bp\right)\mathbb{E}\left[\frac{T-t_m}{T}(a-(1-\alpha)bp)+\frac{T-t_m}{T}\epsilon | a-bp+\epsilon+\frac{T-t_m}{T}\alpha bp \leq q\right] \\
&= F\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)\cdot \\
&\left(\frac{T-t_m}{T}(a-(1-\alpha)bp)+\frac{T-t_m}{T}\int_{-\infty}^{q-(a-bp)-\frac{T-t_m}{T}\alpha bp} xf(x)dx\cdot\frac{1}{F\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)}\right) \\
&= \frac{T-t_m}{T}(a-bp+\alpha bp)F\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)+\frac{T-t_m}{T}\int_{-\infty}^{q-(a-bp)-\frac{T-t_m}{T}\alpha bp} xf(x)dx
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\mathbb{E}\left[\min[\max(q-D,0),D']\right] = \left(q-\frac{t_m}{T}(a-bp)\right)F\left(\frac{T}{t_m}q-(a-bp)\right) \\
&-\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)F\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right) \\
&-\frac{t_m}{T}\int_{q-(a-bp)-\frac{T-t_m}{T}\alpha bp}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx+\frac{T-t_m}{T}\int_{-\infty}^{q-(a-bp)-\frac{T-t_m}{T}\alpha bp} xf(x)dx
\end{aligned}
$$

Accordingly,

$$
\begin{aligned}
&\Pi_r(q,t_m) = p\cdot\mathbb{E}\left[\min(q,D)\right] + (1-\alpha)p\cdot\mathbb{E}\left[\min((q-D)^+,D')\right] - (c_r+w)q \\
&= p\frac{t_m}{T}(a-bp)F\left(\frac{T}{t_m}q-(a-bp)\right) + p\frac{t_m}{T}\int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx + pq\left(1-F\left(\frac{T}{t_m}q-(a-bp)\right)\right) \\
&+ (1-\alpha)[p\left(q-\frac{t_m}{T}(a-bp)\right)F\left(\frac{t_m}{T}q-(a-bp)\right) - \left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)\cdot \\
&F\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right) - \frac{t_m}{T}\int_{q-(a-bp)-\frac{T-t_m}{T}\alpha bp}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx \\
&+ \frac{T-t_m}{T}\int_{-\infty}^{q-(a-bp)-\frac{T-t_m}{T}\alpha bp} xf(x)dx] - (c_r+w)q
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial\Pi_r(q,t_m)}{\partial q} = p(a-bp)f\left(\frac{T}{t_m}q-(a-bp)\right) + p\left(\frac{T}{t_m}q-(a-bp)\right) + p\left(1-F\left(\frac{T}{t_m}q-(a-bp)\right)\right) \\
&- pq\frac{T}{t_m}f\left(\frac{T}{t_m}q-(a-bp)\right) + (1-\alpha)p\left(F\left(\frac{T}{t_m}q-(a-bp)\right)\right) + \left(\frac{T}{t_m}q-(a-bp)\right)\cdot \\
&f\left(\frac{T}{t_m}q-(a-bp)\right) - F\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right) - \left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)\cdot \\
&f\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right) - \left(\frac{T}{t_m}q-(a-bp)\right)f\left(\frac{T}{t_m}q-(a-bp)\right) \\
&+ \frac{t_m}{T}\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)f\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right) + \\
&\frac{T-t_m}{T}\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)f\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right) - (c_r+w) \\
&= p\left(1-F\left(\frac{T}{t_m}q-(a-bp)\right)\right) \\
&+ (1-\alpha)p\left(F\left(\frac{T}{t_m}q-(a-bp)\right) - F\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)\right) - (c_r+w)
\end{aligned}
$$

$$
\frac{\partial^2\Pi_r(q,t_m)}{\partial q^2} = -\frac{T}{t_m}\alpha pf\left(\frac{T}{t_m}q-(a-bp)\right) - (1-\alpha)pf\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right) < 0
$$

$\because$ the probability density function $f(\cdot)$ is always nonnegative. Q.E.D.

A.2

$$\frac{\partial \Pi_r(q,t_m)}{\partial q} = p\left(1 - F\left(\frac{T}{t_m}q - (a-bp)\right)\right)$$
$$+ (1-\alpha)p\left(F\left(\frac{T}{t_m}q - (a-bp)\right) - F\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right)\right) - (c_r + w)$$
$$\iff \alpha F\left(\frac{T}{t_m}q^* - (a-bp)\right) + (1-\alpha)F\left(q^* - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) = \frac{p-(c_r+w)}{p}$$

Because the range of $F(\cdot)$ is between 0 and 1 and its value varies according to the q^*, $\frac{p-(c_r+w)}{p}$ can be expressed as a convex combination of $F\left(\frac{T}{t_m}q^* - (a-bp)\right)$ and $F\left(q^* - (a-bp) - \frac{T-t_m}{T}\alpha bp\right)$. Thus, there exists a unique q^* maximizing the expected profit function of the retailer Π_r. Also, $F\left(\frac{T}{t_m}q^* - (a-bp)\right)$ is always larger than $F\left(q^* - (a-bp) - \frac{T-t_m}{T}\alpha bp\right)$ because $\frac{T}{t_m}q - (a-bp) \geq q - (a-bp) - \frac{T-t_m}{T}\alpha bp$ holds true and $F(\cdot)$ has the non-decreasing property. Q.E.D.

A.3

$$\frac{\partial \Pi_r(q,t_m)}{\partial t_m} = \frac{p}{T}(a-bp)F\left(\frac{T}{t_m}q - (a-bp)\right) + \frac{p}{T}\int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx$$
$$+ (1-\alpha)p[-\frac{1}{T}(a-bp)F\left(\frac{T}{t_m}q - (a-bp)\right) - \frac{\alpha bp}{T}F\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right)$$
$$-\frac{1}{T}\int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx] = \frac{\alpha p}{T}[(a-bp)F\left(\frac{T}{t_m}q - (a-bp)\right)$$
$$-(1-\alpha)bpF\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) + \int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx]$$

$$\frac{\partial^2 \Pi_r(q,t_m)}{\partial t_m^2} = \frac{\alpha p}{T}[-\frac{qT}{t_m^2}(a-bp)f\left(\frac{T}{t_m}q-(a-bp)\right) - \frac{\alpha b^2 p^2}{T}(1-\alpha)\cdot$$
$$f\left(q-(a-bp)-\frac{T-t_m}{t_m}\alpha bp\right) - \frac{qT}{t_m^2}\left(\frac{T}{t_m}q-(a-bp)\right)f\left(\frac{T}{t_m}q-(a-bp)\right)]$$
$$= -\frac{\alpha p q^2 T}{t_m^3}f\left(\frac{T}{t_m}q-(a-bp)\right) - \left(\frac{\alpha bp}{T}\right)^2(1-\alpha)pf\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right) < 0$$

A.4

Let $\mathbb{H}$ be a Hessian matrix of $\Pi_r(q,t_m)$ as follows: $\mathbb{H} = \begin{bmatrix} \frac{\partial^2 \Pi_r(q,t_m)}{\partial q^2} & \frac{\partial^2 \Pi_r(q,t_m)}{\partial q \partial t_m} \\ \frac{\partial^2 \Pi_r(q,t_m)}{\partial t_m \partial q} & \frac{\partial^2 \Pi_r(q,t_m)}{\partial t_m^2} \end{bmatrix}$. By using the second partial derivative test, it is easy to show that the determinant of the Hessian matrix, $\mathbb{D}$, is larger than zero.

$$\mathbb{D} = \left[-\frac{T}{t_m}\alpha pf\left(\frac{T}{t_m}q-(a-bp)\right) - (1-\alpha)pf\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)\right]\cdot$$
$$\left[-\frac{\alpha p q^2 T}{{t_m}^3}f\left(\frac{T}{t_m}q-(a-bp)\right) - \left(\frac{\alpha bp}{T}\right)^2(1-\alpha)pf\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)\right]$$
$$-\left(\frac{\alpha pqT}{{t_m}^2}f\left(\frac{T}{t_m}q-(a-bp)\right) - \frac{1}{T}f\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)\right)^2$$

$\mathbb{D}$ can be summarized as follows:

$$\mathbb{D} = \mathcal{A}\cdot f\left(\frac{T}{t_m}q-(a-bp)\right)^2 + \mathcal{B}\cdot f\left(\frac{T}{t_m}q-(a-bp)\right)f\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)$$
$$+\mathcal{C}\cdot f\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)^2$$

Coefficient $\mathcal{A}$ is canceled to zero. In the case of $\mathcal{B}$, all coefficients have non-negative conditions. Coefficient $\mathcal{C}$ consists of $\left(((1-\alpha)\cdot\alpha)^2 p^2\left(\frac{bp}{T}\right)^2 - \frac{1}{T^2}\right)$. The first term is always non-negative. The second term can be offset by part of the coefficient of $\mathcal{B}$, $\frac{2\alpha pq}{{t_m}^2}f\left(\frac{T}{t_m}q-(a-bp)\right)f\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)$. Since inequalities $pq \gg \alpha$, $T \geq$

t_m, and $2f\left(\frac{T}{t_m}q-(a-bp)\right) \geq f\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)$ hold true, coefficient $\mathcal{C}$ is non-negative. Accordingly, the Hessian matrix $\mathbb{H}$ is negative definite and the profit function has the unique combination (t_m^{**}, q^{**}), which is the maximizer of the function. Q.E.D.

A.5

As shown in Observation 3, the manufacturer's maximum profit occurs when the retailer starts the markdown sale at $t_m = 0$. Consider the condition that the expected profit function of the retailer decreases with increasing t_m that the first derivative of Π_r by t_m is negative.

$$\frac{\partial \Pi_r(q,t_m)}{\partial t_m} = \frac{\alpha p}{T}[(a-bp)F\left(\frac{T}{t_m}q-(a-bp)\right)-(1-\alpha)bpF\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)$$
$$+\int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx]$$

Recall Equation (3),

$$\alpha F\left(\frac{T}{t_m}q^*-(a-bp)\right)+(1-\alpha)F\left(q^*-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)=\frac{p-(c_r+w)}{p}$$

When the retailer places an optimal order quantity q^* from her standpoint, the above equation holds. By substituting the second term of the left-hand side with the first derivative of the profit function by t_m, it can be represented as follows:

$$\frac{\alpha p}{T}\left((a-bp+\alpha bp)F\left(\frac{T}{t_m}q^*-(a-bp)\right)-b(p-c_r-w)+\int_{-\infty}^{\frac{T}{t_m}q^*-(a-bp)} xf(x)dx\right)$$

Let the first derivative of the expected profit function with respect to t_m becomes negative.

$$(a - bp + \alpha bp)F\left(\frac{T}{t_m}q^* - (a - bp)\right) - b(p - c_r - w) + \int_{-\infty}^{\frac{T}{t_m}q^* - (a-bp)} xf(x)dx < 0$$

The above inequality holds true even when the maximum value of the left-hand side has a negative value. To show the maximum value of the left-hand side, let q^* as the upper bound and t_m as the limit to 0.

$$a - bp + \alpha bp - bp + bc_r + bw < 0$$

For the wholesale price w, a sufficient condition in Corollary 2 can be shown.

$$w < 2p - c_r - \alpha p - \frac{a}{b} \text{ Q.E.D.}$$

A.6

Let the first derivative of the expected profit function with respect to t_m becomes positive when t_m equals to T.

$$(a - bp + \alpha bp)F\left(\frac{T}{t_m}q^* - (a - bp)\right) - b(p - c_r - w) + \int_{-\infty}^{q^* - (a-bp)} xf(x)dx > 0$$

When t_m equals to T, q^* equals to $F^{-1}1\left(\frac{p-(c_r+w)}{p}\right) + a - bp$ by the Corollary 1. Thus, the following inequality holds true which is the sufficient condition shown in

Corollary 3.

$$(a - bp + \alpha bp)\left(\frac{p - (c_r + w)}{p}\right) - b(p - c_r - w) + \int_{-\infty}^{F^{-1}\left(\frac{p-(c_r+w)}{p}\right)} xf(x)dx > 0 \text{ Q.E.D.}$$

Appendix B

B.1

Consider the problem under the planning horizon $t \in \{1, \ldots, 5\}$. Assume an expiry date, τ, as 2 and initial inventory levels, u_1 and v_1, as 0. Table B.1 summarizes the relevant costs of this numerical example. For the capacity of the order quantity, K_t was assumed as 30 for every period. Purchasing demands were assumed as $\mathbf{d}_t = (11, 18, 15, 19)$. Consequently, revisiting demands $\tilde{\xi}_t$ were developed as follows:

$$\tilde{\xi}_1(\tilde{\boldsymbol{\rho}}) = 11\tilde{\rho}_1^1$$

$$\tilde{\xi}_2(\tilde{\boldsymbol{\rho}}) = 11\tilde{\rho}_1^2 + 18\tilde{\rho}_2^2$$

$$\tilde{\xi}_3(\tilde{\boldsymbol{\rho}}) = 18\tilde{\rho}_2^3 + 15\tilde{\rho}_3^3$$

$$\tilde{\xi}_4(\tilde{\boldsymbol{\rho}}) = 15\tilde{\rho}_3^4 + 19\tilde{\rho}_4^4$$

We made the decision variables based on the LDR as follows:

$$\mathbf{x_t}(\tilde{\boldsymbol{\rho}}) = \{x_1^0, x_2^0 + x_2^{1,1}\tilde{\rho}_1^1, x_3^0 + x_3^{1,1}\tilde{\rho}_1^1 + x_3^{1,2}\tilde{\rho}_1^2 + x_3^{2,2}\tilde{\rho}_2^2, x_4^0 + x_4^{1,1}\tilde{\rho}_1^1 + x_4^{1,2}\tilde{\rho}_1^2 + x_4^{2,2}\tilde{\rho}_2^2 + x_4^{2,3}\tilde{\rho}_2^3 + x_4^{3,3}\tilde{\rho}_3^3\}$$

$$\mathbf{y_t}(\tilde{\boldsymbol{\rho}}) = \{y_1^0, y_2^0 + y_2^{1,1}\tilde{\rho}_1^1, y_3^0 + y_3^{1,1}\tilde{\rho}_1^1 + y_3^{1,2}\tilde{\rho}_1^2 + y_3^{2,2}\tilde{\rho}_2^2, y_4^0 + y_4^{1,1}\tilde{\rho}_1^1 + y_4^{1,2}\tilde{\rho}_1^2 + y_4^{2,2}\tilde{\rho}_2^2 + y_4^{2,3}\tilde{\rho}_2^3 + y_4^{3,3}\tilde{\rho}_3^3\}$$

Accordingly, decision variables for the inventory levels also take the affine function

Table B.1: Related costs of the numerical example

	Planning horizon t			
Cost per unit	1	2	3	4
c_t	10	10	10	10
h_t	1.45	1.67	1.90	1.57
b_t	4.78	4.92	4.17	4.35
p_t	47.78	49.18	41.65	43.45

of uncertainty factors as follows:

$$\mathbf{u_t}(\tilde{\boldsymbol{\rho}}) = \big\{u_1^0, u_2^0 + u_2^{1,1}\tilde{\rho}_1^1, u_3^0 + u_3^{1,1}\tilde{\rho}_1^1 + u_2^{1,2}\tilde{\rho}_1^2 + u_3^{2,2}\tilde{\rho}_2^2, u_4^0 + u_4^{1,1}\tilde{\rho}_1^1 + u_4^{1,2}\tilde{\rho}_1^2 + u_4^{2,2}\tilde{\rho}_2^2 + u_4^{2,3}\tilde{\rho}_2^3 + u_4^{3,3}\tilde{\rho}_3^3$$
$$+ u_5^{1,1}\tilde{\rho}_1^1 + u_5^{1,2}\tilde{\rho}_1^2 + u_5^{2,2}\tilde{\rho}_2^2 + u_5^{2,3}\tilde{\rho}_2^3 + u_5^{3,3}\tilde{\rho}_3^3 + u_5^{3,4}\tilde{\rho}_3^4 + u_5^{4,4}\tilde{\rho}_4^4\big\}$$

$$\mathbf{v_t}(\tilde{\boldsymbol{\rho}}) = \big\{v_1^0, v_2^0 + v_2^{1,1}\tilde{\rho}_1^1, v_3^0 + v_3^{1,1}\tilde{\rho}_1^1 + v_2^{1,2}\tilde{\rho}_1^2 + v_3^{2,2}\tilde{\rho}_2^2, v_4^0 + v_4^{1,1}\tilde{\rho}_1^1 + v_4^{1,2}\tilde{\rho}_1^2 + v_4^{2,2}\tilde{\rho}_2^2 + v_4^{2,3}\tilde{\rho}_2^3 + v_4^{3,3}\tilde{\rho}_3^3$$
$$+ v_5^{1,1}\tilde{\rho}_1^1 + v_5^{1,2}\tilde{\rho}_1^2 + v_5^{2,2}\tilde{\rho}_2^2 + v_5^{2,3}\tilde{\rho}_2^3 + v_5^{3,3}\tilde{\rho}_3^3 + v_5^{3,4}\tilde{\rho}_3^4 + v_5^{4,4}\tilde{\rho}_4^4\big\}$$

We will derive the balance equations concerning $\mathbf{y}$, $\mathbf{v}$, and $\tilde{\boldsymbol{\xi}}$ while omitting the balance equations for $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{d}$ which is easy to show. Balance equations for $\mathbf{y}$,

$\mathbf{v}$, and $\tilde{\boldsymbol{\xi}}$ for the entire periods $t \in \{1, \ldots, 5\}$ are derived in (B.1) as follows:

$$\begin{aligned}
&v_2^0 + v_2^{1,1}\tilde{\rho}_1^1 = v_1^0 + y_1^0 - 11\tilde{\rho}_1^1 \\
&\iff \begin{cases} v_2^0 = v_1^0 + y_1^0 \\ v_2^{1,1} = -11 \end{cases} \\
&v_3^0 + v_3^{1,1}\tilde{\rho}_1^1 + v_3^{1,2}\tilde{\rho}_1^2 + v_3^{2,2}\tilde{\rho}_2^2 = v_2^0 + v_2^{1,1}\tilde{\rho}_1^1 + y_2^0 + y_2^{1,1}\tilde{\rho}_1^1 - 11\tilde{\rho}_1^2 - 18\tilde{\rho}_2^2 \\
&\iff \begin{cases} v_3^0 = v_2^0 + y_2^0 \\ v_3^{1,1} = v_2^{1,1} + y_2^{1,1} \\ v_3^{1,2} = -11 \\ v_3^{2,2} = -18 \end{cases} \\
&v_4^0 + v_4^{1,1}\tilde{\rho}_1^1 + v_4^{1,2}\tilde{\rho}_1^2 + v_4^{2,2}\tilde{\rho}_2^2 + v_4^{2,3}\tilde{\rho}_2^3 + v_4^{3,3}\tilde{\rho}_3^3 \\
&= v_3^0 + v_3^{1,1}\tilde{\rho}_1^1 + v_3^{1,2}\tilde{\rho}_1^2 + v_3^{2,2}\tilde{\rho}_2^2 + y_3^0 + y_3^{1,1}\tilde{\rho}_1^1 + y_3^{1,2}\tilde{\rho}_1^2 + y_3^{2,2}\tilde{\rho}_2^2 - 18\tilde{\rho}_2^3 - 15\tilde{\rho}_3^3 \\
&\iff \begin{cases} v_4^{1,1} = v_3^{1,1} + y_3^{1,1} \\ v_4^{1,2} = v_3^{1,2} + y_3^{1,2} \\ v_4^{2,2} = v_3^{2,2} + y_3^{2,2} \\ v_4^{2,3} = -18 \\ v_4^{3,3} = -15 \end{cases}
\end{aligned} \tag{B.1}$$

$$
\begin{aligned}
&v_5^0 + v_5^{1,1}\tilde{\rho}_1^1 + v_5^{1,2}\tilde{\rho}_1^2 + v_5^{2,2}\tilde{\rho}_2^2 + v_5^{2,3}\tilde{\rho}_2^3 + v_5^{3,3}\tilde{\rho}_3^3 + v_5^{3,4}\tilde{\rho}_3^4 + v_5^{4,4}\tilde{\rho}_4^4 \\
&= v_4^0 + v_4^{1,1}\tilde{\rho}_1^1 + v_4^{1,2}\tilde{\rho}_1^2 + v_4^{2,2}\tilde{\rho}_2^2 + v_4^{2,3}\tilde{\rho}_2^3 + v_4^{3,3}\tilde{\rho}_3^3 + y_4^0 + y_4^{1,1}\tilde{\rho}_1^1 + y_4^{1,2}\tilde{\rho}_1^2 + y_4^{2,2}\tilde{\rho}_2^2 + y_4^{2,3}\tilde{\rho}_2^3 + y_4^{3,3}\tilde{\rho}_3^3 \\
&\quad - 15\rho_3^4 - 19\tilde{\rho}_4^4 \\
&\iff \begin{cases}
v_5^0 = v_4^0 + y_4^0 \\
v_5^{1,1} = v_4^{1,1} + y_4^{1,1} \\
v_5^{1,2} = v_4^{1,2} + y_4^{1,2} \\
v_5^{2,2} = v_4^{2,2} + y_4^{2,2} \\
v_5^{2,3} = v_4^{2,3} + y_4^{2,3} \\
v_5^{3,3} = v_4^{3,3} + y_4^{3,3} \\
v_5^{3,4} = -15 \\
v_5^{4,4} = -19
\end{cases}
\end{aligned}
$$

Based on the (B.1), the solution, which is the inventory policy, could be obtained by solving the robust counterpart as follows:

$$
\begin{aligned}
&\mathbf{x}_t(\tilde{\boldsymbol{\rho}}) = \{16.12, 14.88, 15, 11\} \\
&\mathbf{y}_t(\tilde{\boldsymbol{\rho}}) = \left\{13.88, 15.12, 9.88 + 5.12\tilde{\rho}_1^1 + 5.12\tilde{\rho}_1^2, 3.73 + 4.13\tilde{\rho}_1^1 + 4.13\tilde{\rho}_1^2 + 11.14\tilde{\rho}_2^2 + 11.14\tilde{\rho}_2^3\right\}
\end{aligned}
$$

B.2

We generated a random value from a uniform distribution with support [0, 1] for all non-zero $\tilde{\boldsymbol{\rho}}$. Afterward, all $\tilde{\boldsymbol{\rho}}$ were normalized so that the sum in the same row became 1. For the last elements in each row, we forced the sum of the revisiting rate to be 1. The pseudocode is described in Algorithm 1.

Algorithm 2 Generation of uncertainty factors

---

while $i, j \in \mathfrak{T}^-$ **do**
    if $j \geq i$ **then**
        $\tilde{\rho}_i^j \leftarrow$ uniform $(0,\ 1)$
    end
end
while $i, j \in \mathfrak{T}^-$ **do**
    $\tilde{\rho}_i^j \leftarrow \tilde{\rho}_i^j / \sum_{j \in \mathfrak{T}^-} \tilde{\rho}_i^j$
end

B.3

For IC, we generated the random value based on the uniform distribution whose support is [0, 1]. In succession, the next generated random value has the support between zero and the value obtained by subtracting the cumulative sum of generated random values from 1. We proceeded recursively in this manner and forced the sum from the purchasing date to the last revisiting date to be 1. In the case of data generation for PC, IC data was generated and rearranged in the reverse order at the same row. The pseudocodes of IC and PC are described in Algorithms 2 and 3, respectively.

Algorithm 3 Generation of uncertainty factors IC

---

while $i, j \in \mathfrak{T}^-$ **do**
    if $j \geq i$ **then**
        $\tilde{\rho}_i^j \leftarrow$ uniform $(0,\ 1 - \sum_{k=1}^{j-1} \tilde{\rho}_i^j)$
    end
end

Algorithm 4 Generation of uncertainty factors PC

while $i, j \in \mathfrak{T}^-$ **do**
  if $j \geq i$ **then**
    $\tilde{\rho}_i^j \leftarrow$ uniform $(0,\, 1 - \sum_{k=1}^{j-1} \tilde{\rho}_i^j)$
  end
end
while $i, j \in \mathfrak{T}^-$ **do**
  while $k \in \{1, \cdots, \tau\}$ **do**
    $\tilde{\rho}_i^{i+k-1} \leftarrow \tilde{\rho}_i^{i+\tau-k}$
  end
end

Appendix C

C.1

When a trade-in program is introduced, distinct differences exist from the model, which only considers the refurbishment service. For the previous refurbishment service, customers return end-of-life products, and they are sent to the remanufacturer. After being remanufactured, the refurbished products are sent to the retailer. For the customer who has submitted the end-of-life product, the refurbished product should be provided immediately. If a refurbished product is not available, discontinued old-generation products may be pre-produced to prevent backlog. However, this process produces discontinued products, which incurs a relatively higher cost, including opportunity cost. Also, in the case of out of stock, dissatisfaction with the warranty service will have a significant detrimental effect on the brand's image, which can lead to a high penalty cost for the retailer. Figure C.1 is provided to illustrate the differences from the IMRSTIP in terms of product flow.

C.2

Using the demand models (4.26) and (4.27), we provide an example based on the small data. Assume that the inventory manager has historical data, including three

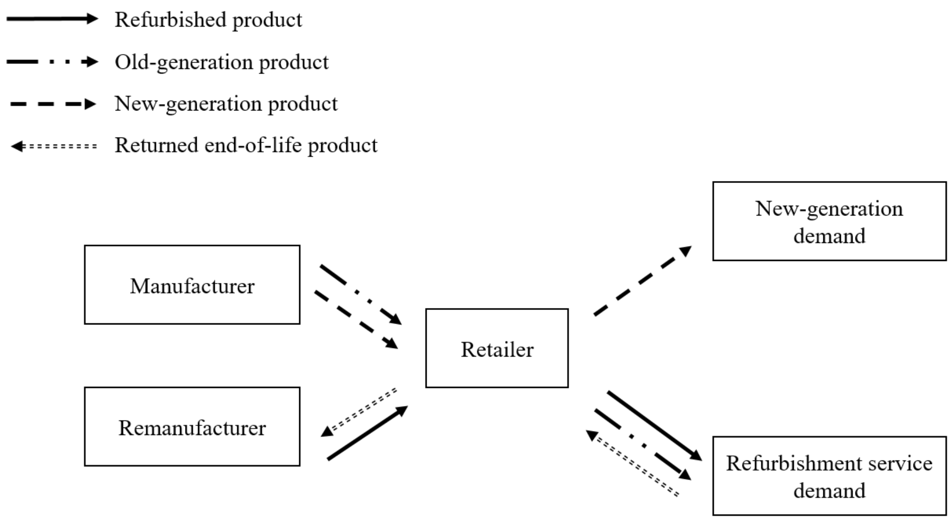


Figure C.1: Flow of products when the trade-in program is not considered

types of demands, $\zeta^-, \phi^-,$ and ξ^-, from the past as follows:

$$\zeta^- = (22, 23, 21, 22, 25, 21, 20, 22, 24)$$

$$\phi^- = (20, 17, 15, 17, 24, 22, 19, 21, 23)$$

$$\psi^- = (17, 14, 16, 15, 14, 15, 16, 14, 13)$$

We assume that the inventory manager establishes the inventory policy from $t = 1$ to $t = 3$. From the VARMA (1, 0) process, three types of demand models were estimated as follows:

$$\begin{aligned}
\tilde{\zeta}_t &= 22.25 + 0.08\zeta_{t-1} - 0.18\phi_{t-1} + 0.11\psi_{t-1} + \epsilon_{1,t} \\
\tilde{\phi}_t &= 40.75 - 0.63\zeta_{t-1} + 0.42\phi_{t-1} - 1.02\psi_{t-1} + \epsilon_{2,t} \\
\tilde{\psi}_t &= 15.38 + 0.11\zeta_{t-1} - 0.06\phi_{t-1} - 0.14\psi_{t-1} + \epsilon_{3,t}
\end{aligned} \tag{C.1}$$

The estimated covariance matrix of residuals, $\boldsymbol{\epsilon}_t$, was obtained as follows:

$$\begin{array}{c c} & \begin{array}{ccc} \epsilon_{1,t} & \epsilon_{2,t} & \epsilon_{3,t} \end{array} \\ \begin{array}{c} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{array} & \begin{pmatrix} 4.39 & 6.39 & -2.60 \\ 6.39 & 14.27 & -3.70 \\ -2.60 & -3.70 & 1.84 \end{pmatrix} \end{array}$$

If residuals $(\epsilon_{1,1}, \epsilon_{1,2}, \epsilon_{1,3}, \epsilon_{2,1}, \epsilon_{2,2}, \cdots)$ were redefined to $(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4, \tilde{z}_5, \cdots)$, the demand model (C.1) could be induced to the factor-based demand model (C.2) as follows:

$$
\begin{aligned}
\tilde{\zeta}_1 &= 21.46 + \tilde{z}_1 \\
\tilde{\phi}_1 &= 22.03 + \tilde{z}_2 \\
\tilde{\psi}_1 &= 14.82 + \tilde{z}_3 \\
\tilde{\zeta}_2 &= 22.25 + 0.08(21.46 + \tilde{z}_1) - 0.18(22.03 + \tilde{z}_2) + 0.11(14.82 + \tilde{z}_3) + \tilde{z}_4 \\
&= 21.63 + 0.08\tilde{z}_1 - 0.18\tilde{z}_2 + 0.11\tilde{z}_3 + \tilde{z}_4 \\
\tilde{\phi}_2 &= 40.75 - 0.63(21.46 + \tilde{z}_1) + 0.42(22.03 + \tilde{z}_2) - 1.02(14.82 + \tilde{z}_3) + \tilde{z}_5 \\
&= 21.37 - 0.63\tilde{z}_1 + 0.42\tilde{z}_2 - 1.02\tilde{z}_3 + \tilde{z}_5 \\
\tilde{\psi}_2 &= 15.38 + 0.11(21.46 + \tilde{z}_1) - 0.06(22.03 + \tilde{z}_2) - 0.14(14.82 + \tilde{z}_3) + \tilde{z}_6 \\
&= 14.34 + 0.11\tilde{z}_1 - 0.06\tilde{z}_2 - 0.14\tilde{z}_3 + \tilde{z}_6 \\
\tilde{\zeta}_3 &= 22.25 + 0.08(21.63 + 0.08\tilde{z}_1 - 0.18\tilde{z}_2 + 0.11\tilde{z}_3 + \tilde{z}_4) \qquad \text{(C.2)} \\
&\quad - 0.18(21.37 - 0.63\tilde{z}_1 + 0.42\tilde{z}_2 - 1.02\tilde{z}_3 + \tilde{z}_5) \\
&\quad + 0.11(14.34 + 0.11\tilde{z}_1 - 0.06\tilde{z}_2 - 0.14\tilde{z}_3 + \tilde{z}_6) + \tilde{z}_7 \\
&= 21.71 + 0.13\tilde{z}_1 - 0.10\tilde{z}_2 + 0.18\tilde{z}_3 + 0.08\tilde{z}_4 - 0.18\tilde{z}_5 + 0.11\tilde{z}_6 + \tilde{z}_7 \\
\tilde{\phi}_3 &= 40.75 - 0.63(21.63 + 0.08\tilde{z}_1 - 0.18\tilde{z}_2 + 0.11\tilde{z}_3 + \tilde{z}_4) \\
&\quad + 0.42(21.37 - 0.63\tilde{z}_1 + 0.42\tilde{z}_2 - 1.02\tilde{z}_3 + \tilde{z}_5) \\
&\quad - 1.02(14.34 + 0.11\tilde{z}_1 - 0.06\tilde{z}_2 - 0.14\tilde{z}_3 + \tilde{z}_6) + \tilde{z}_8 \\
&= 21.47 - 0.43\tilde{z}_1 + 0.35\tilde{z}_2 - 0.35\tilde{z}_3 - 0.63\tilde{z}_4 + 0.42\tilde{z}_5 - 1.02\tilde{z}_6 + \tilde{z}_8 \\
\tilde{\psi}_3 &= 15.38 + 0.11(21.63 + 0.08\tilde{z}_1 - 0.18\tilde{z}_2 + 0.11\tilde{z}_3 + \tilde{z}_4) \\
&\quad - 0.06(14.34 + 0.11\tilde{z}_1 - 0.06\tilde{z}_2 - 0.14\tilde{z}_3 + \tilde{z}_6) \\
&\quad - 0.14(14.34 + 0.11\tilde{z}_1 - 0.06\tilde{z}_2 - 0.14\tilde{z}_3 + \tilde{z}_6) + \tilde{z}_9 \\
&= 14.47 + 0.03\tilde{z}_1 - 0.04\tilde{z}_2 + 0.09\tilde{z}_3 + 0.11\tilde{z}_4 - 0.06\tilde{z}_5 + 0.14\tilde{z}_6 + \tilde{z}_9
\end{aligned}
$$

Accordingly, the demand model featured the affine function of the uncertain factors. Based on the derived demand model, the inventory manager established the inventory policy through the robust counterpart (4.19). We assumed that purchasing costs ($c_{n,t}$, $c_{o,t}$, $c_{q,t}$, $c_{m,t}$) were given as (6, 15, 2, 3) for periods $t = 1, 2,$ and 3. For the inventory holding and penalty costs, we assumed that ($h_{u,t}$, $h_{v,t}$, $h_{w,t}$, $h_{I,t}$) and (b_t, p_t) were given as (1.5, 1.2, 1, 1) and (10, 20), respectively, for periods $t = 1, 2,$ and 3. For the capacities of the manufacturer and remanufacturer, C_t and U_t, we assumed that each capacity was 60 and 40 for periods $t = 1, 2,$ and 3. For simplicity, we assumed that all lead times are zero. Based on the given costs and parameters, the inventory manager established the policy as follows:

$$
\begin{aligned}
x_{n,t}(\tilde{\boldsymbol{z}}) &= (56.67, 33.45 + 1.45\tilde{z}_1 + 1.24\tilde{z}_2 - 0.91\tilde{z}_3, 43.18 + 0.03\tilde{z}_1 + 0.38\tilde{z}_2 - 0.60\tilde{z}_3 + 0.12\tilde{z}_4 \\
&\qquad + 0.11\tilde{z}_5 - 0.65\tilde{z}_6) \\
y_{o,t}(\tilde{\boldsymbol{z}}) &= (0, 0, 0) \\
q_t(\tilde{\boldsymbol{z}}) &= (17.16, 11.99 + 0.11\tilde{z}_1 - 0.06\tilde{z}_2 + 0.86\tilde{z}_3, 14.47 + 0.07\tilde{z}_1 - 0.05\tilde{z}_2 - 0.03\tilde{z}_3 + 0.08\tilde{z}_4 \\
&\qquad - 0.05\tilde{z}_5 - 0.02\tilde{z}_6) \\
m_t(\tilde{\boldsymbol{z}}) &= (0, 0, 0)
\end{aligned}
$$

In this example, the order quantity for the remanufacturing process was not featured. The demand of the trade-in program was relatively high and sufficient end-of-use products were returned to the retailer. Since the refurbishing cost is less than the remanufacturing cost, the retailer did not send the end-of-life products to the remanufacturer. Also, refurbishment service did not encounter the backlog. As a result, the manufacturer did not produce the old-generation products. That is, the inventory policies for $y_t(\tilde{\boldsymbol{z}})$ and $m_t(\tilde{\boldsymbol{z}})$ were zero for all periods.

국문초록

시장의 세계화에 따른 기업 간의 경쟁이 가속화됨에 따라, 단기 인센티브를 통해 고객의 제품 또는 서비스 구매를 유도하는 판매촉진의 역할이 중요해졌다. 가격 인하, 행사상품 증정, 트레이드인프로그램과 같은 다양한 유형의 판매촉진 전략이 존재하지만, 공통된 주요 목적은 기업이 고객에게 혜택을 제공하여 고객의 수요를 증대시키는 것이다. 판매촉진의 성공적인 전략은 경영과학 또는 공급망관리 분야를 포함한 관련 학계의 관심을 이끌었다. 지속적인 운영을 위한 전략을 검토하고 전략적 수준 계획을 기반으로 하는 판매 촉진의 효과를 입증하기 위한 다양한 연구가 수행되었습니다. 하지만 운영 수준의 소매업체 입장에서의 연구는 미흡한 실정이다.

본 논문에서는 (i) 마크 다운 (ii) buy one get one free (BOGO), 및 (iii) 트레이드인프로그램을 고려한 재고관리모형을 다룬다. 먼저, 신문가판원 모형에 마크 다운 시작 시점을 나타내는 결정 변수를 도입하여 최적의 마크 다운 시작 시점과 주문량의 조합을 제공하는 모형을 제안한다. 분산 시스템의 특정 조건에서는 소매업자가 가장 높은 이익을 얻는 시점이 제조업자에게 낮은 수익성을 야기할 수 있다. 따라서 본 연구는 제조업자에 대한 소매업자의 비합리적 주문을 막기 위한 이익분배계약을 제안한다. 이익분배계약을 통한 중앙집권화 시스템은 분산 시스템에서 얻은 이익에 비해 소매업자와 제조업자의 이익을 향상시킴을 수치실험을 통해 확인하였다. 둘째, 모바일 어플리케이션 “나만의 냉장고”를 고려한 재고모형을 고려한다. 이 앱을 통해 BOGO 행사제품을 구매한 고객은 증정품을 구매 당일 날 가져가지 않고 미래에 재방문하여 수령할 수 있는 혜택을 받는다. 하지만 소매업자 입장에서는 고객이 증정품을 언제 수령해 갈 지에 대한 불확실성이 존재하며 이는 기존의 재고관리 운영방식에는 한계점이 있음을 시사한다. 본 연구에서는 고객의 재방문에 대한 불확실성을 고려한 복수기간 추계계획 재고모형

을 수립하며 이를 효율적으로 계산하기 위한 강건최적화 모형으로 근사화하였다. 셋째, 리퍼서비스와 트레이드인프로그램을 고려한 폐회로 공급망 시스템 기반의 복수기간 재고관리모형을 제안한다. 신세대 제품, 리퍼서비스 및 트레이드인프로그램에 대한 세 가지 유형의 불확실한 수요에 대한 상관관계를 반영함에 따라 복수기간 추계계획 재고 모형이 수립된다. 복수기간 추계계획 재고모형의 계산이 어렵다는 한계를 극복하고자 강건최적화 모형으로 근사화하였다.

주요어: 판매촉진, 신문가판원 모형, 재고관리 모형, 이익분배계약, 강건 최적화, 분포 강건 최적화

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